3.6. PARCIJALNI IZVODI VIŠEG REDA

Zadatak 1. *Naći mešovite parcijalne izvode funkcije*

$$f(x,y)=\begin{cases} x^2\operatorname{arctg}\frac{y}{x}-y^2\operatorname{arctg}\frac{x}{y}\,, & \textit{ako je } x\neq 0 \textit{ i } y\neq 0\,,\\ 0\,, & \textit{ako je } x=0 \textit{ ili } y=0\,.\end{cases}$$

Rešenje: Ako je $x\neq 0$ i $y\neq 0$, tada je

$$f_x'(x,y)=2x\cdot\operatorname{arctg}\frac{y}{x}-\frac{x^2y}{x^2+y^2}-\frac{y^3}{x^2+y^2}=2x\cdot\operatorname{arctg}\frac{y}{x}-y\,,$$

$$f_{yx}''(x,y)=\frac{2x^2}{x^2+y^2}-1=\frac{x^2-y^2}{x^2+y^2}\,,$$

$$f_y'(x,y)=\frac{x^3}{x^2+y^2}-2y\cdot\operatorname{arctg}\frac{x}{y}+\frac{x^2y}{x^2+y^2}=x-2y\cdot\operatorname{arctg}\frac{x}{y}\,,$$

$$f_{xy}''(x,y)=1-\frac{2y^2}{x^2+y^2}=\frac{x^2-y^2}{x^2+y^2}\,.$$

Dakle, ako je $x\neq 0$ i $y\neq 0$, tada je $f_{xy}''(x,y)=f_{yx}''(x,y)$. Neka je sada $x=0$ i $y\neq 0$. Kako je $f(0,y)=0$, to je

$$f_x'(0,y)=\lim_{x\to 0}\frac{f(x,y)-f(0,y)}{x}=$$

$$=\lim_{x\to 0}\frac{x^2\operatorname{arctg}\frac{y}{x}-y^2\operatorname{arctg}\frac{x}{y}}{x}=\lim_{x\to 0}\left(x\operatorname{arctg}\frac{y}{x}-y\frac{\operatorname{arctg}\frac{x}{y}}{\frac{x}{y}}\right)\,.$$

Osim toga je $\left|\operatorname{arctg}\frac{y}{x}\right| < \frac{\pi}{2}$ i $\operatorname{arctg}\frac{x}{y} \sim \frac{x}{y}$ kada $x \to 0$, pa je $f'_x(0,y) = -y^2 \cdot \frac{1}{y} = -y$. Na sličan način dobijamo da je $f'_y(x,0) = x$. Stoga je $f'_x(0,0) = f'_y(0,0) = 0$. No onda je

$$f''_{yx}(0,0) = \lim_{y\to 0} \frac{f'_x(0,y) - f'_x(0,0)}{y} = \lim_{y\to 0} \frac{-y-0}{y} = -1\,,$$
$$f''_{xy}(0,0) = \lim_{x\to 0} \frac{f'_y(x,0) - f'_y(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1\,,$$

čime smo dokazali da su mešoviti parcijalni izvodi drugog reda međusobno različiti u tački $(0,0)$. Primetimo da su parcijalni izvodi drugog reda prekidne funkcije u toj tački. Zaista, ako uočimo nizove $(x_n, y_n) = (1/n, 1/n)$ i $(x'_n, y'_n) = (2/n, 1/n)$, onda je $\lim(x_n, y_n) = \lim(x'_n, y'_n) = (0,0)$, ali je s druge strane

$$\lim_{n\to+\infty} f''_{xy}(x_n, y_n) = \lim_{n\to+\infty} f''_{yx}(x_n, y_n) = \lim_{n\to+\infty} \frac{\frac{1}{n^2} - \frac{1}{n^2}}{\frac{1}{n^2} + \frac{1}{n^2}} = 0\,,$$
$$\lim_{n\to+\infty} f''_{xy}(x'_n, y'_n) = \lim_{n\to+\infty} f''_{yx}(x'_n, y'_n) = \lim_{n\to+\infty} \frac{\frac{4}{n^2} - \frac{1}{n^2}}{\frac{4}{n^2} + \frac{1}{n^2}} = \frac{3}{5}\,,$$

pa su drugi mešoviti izvodi prekidne funkcije u tački $(0,0)$.

Zadatak 2. *Naći parcijalne izvode prvog i drugog reda funkcije $z = z(u,v)$, gde je $u = x^2 + y^2$, $v = xy$, pretpostavljajući da je funkcija z klase $C^{(2)}(D)$ u nekoj oblasti D iz $\mathbb{R}^2$.*

Rešenje: Parcijalni izvodi prvog reda su:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = 2x \cdot \frac{\partial z}{\partial u} + y \cdot \frac{\partial z}{\partial v}\,,$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = 2y \cdot \frac{\partial z}{\partial u} + x \cdot \frac{\partial z}{\partial v}\,.$$

Da bi smo odredili druge parcijalne izvode, odredimo izvod funkcije $\frac{\partial z}{\partial x}$ po x i y, kao i izvod funkcije $\frac{\partial z}{\partial y}$ po y. Pri tome treba imati u vidu činjenicu da su $\frac{\partial z}{\partial u}$ i $\frac{\partial z}{\partial v}$ funkcije promenljivih x i y posredstvom funkcija u i v. Tako dobijamo

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x \cdot \frac{\partial z}{\partial u} + y \cdot \frac{\partial z}{\partial v}\right) = \\
&= 2x \cdot \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + 2 \cdot \frac{\partial z}{\partial u} + y \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) + 0 \cdot \frac{\partial z}{\partial v} = \\
&= 2x\left(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v \partial u}\frac{\partial v}{\partial x}\right) + 2 \cdot \frac{\partial z}{\partial u} + y\left(\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\right) = \\
&= 2x\left(2x\frac{\partial^2 z}{\partial u^2} + y\frac{\partial^2 z}{\partial v \partial u}\right) + 2 \cdot \frac{\partial z}{\partial u} + y\left(2x\frac{\partial^2 z}{\partial u \partial v} + y\frac{\partial^2 z}{\partial v^2}\right) .
\end{aligned}$$

Mešoviti parcijalni izvodi drugog reda funkcije $z(u, v)$ su jednaki zbog pretpostavljene neprekidnosti, pa je

$$\frac{\partial^2 z}{\partial x^2} = 4x^2\frac{\partial^2 z}{\partial u^2} + 4xy\frac{\partial^2 z}{\partial u \partial v} + y^2\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial z}{\partial u} .$$

Analogno dobijamo

$$\begin{aligned}
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(2y \cdot \frac{\partial z}{\partial u} + x \cdot \frac{\partial z}{\partial v}\right) = \\
&= 2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + 2\frac{\partial z}{\partial u} + x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) + 0\,\frac{\partial z}{\partial v} = \\
&= 2y\left(2y\frac{\partial^2 z}{\partial u^2} + x\frac{\partial^2 z}{\partial v \partial u}\right) + 2\frac{\partial z}{\partial u} + x\left(2y \cdot \frac{\partial^2 z}{\partial u \partial v} + x \cdot \frac{\partial^2 z}{\partial v^2}\right) + 0 \cdot \frac{\partial z}{\partial v} = \\
&= 4y^2 \cdot \frac{\partial^2 z}{\partial u^2} + 4xy \cdot \frac{\partial^2 z}{\partial u \partial v} + x^2 \cdot \frac{\partial^2 z}{\partial v^2} + 2 \cdot \frac{\partial z}{\partial u} .
\end{aligned}$$

Mešoviti parcijalni izvodi drugog reda su

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x \cdot \frac{\partial z}{\partial u} + y \cdot \frac{\partial z}{\partial v}\right) = \\
&= 2x \cdot \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial z}{\partial u} \cdot \frac{\partial}{\partial y}(2x) + y \cdot \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) + \frac{\partial z}{\partial v} \cdot \frac{\partial}{\partial y}(y) = \\
&= 2x\left(2y\frac{\partial^2 z}{\partial u^2} + x\frac{\partial^2 z}{\partial v \partial u}\right) + 0\frac{\partial z}{\partial u} + y\left(2y\frac{\partial^2 z}{\partial u \partial v} + x\frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial v} = \\
&= 4xy\frac{\partial^2 z}{\partial u^2} + 2(x^2 + y^2)\frac{\partial^2 z}{\partial u \partial v} + xy\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v} .
\end{aligned}$$

Smena promenljivih

1. *Smena promenljivih u izrazima koji sadrže obične izvode*
Neka je dat izraz

$$\Phi = F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\right),$$

koji sadrži nezavisno promenljivu x, funkciju $y(x)$ i njene izvode proizvoljnog reda. Ako je potrebno transformisati ovaj izraz uvođenjem nove nezavisno promenljive t i nove funkcije $u = u(t)$, koje su sledećim jednakostima:

$$(1) \qquad x = f(t,u), \quad y = g(t,u),$$

povezane sa promenljivima x i y, onda iz (1) dobijamo

$$(2) \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{\partial g}{\partial t} + \frac{\partial g}{\partial u}\cdot\frac{du}{dt}}{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial u}\cdot\frac{du}{dt}}, \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}, \quad \text{itd.}$$

Korišćenjem (1) i (2), dati izraz dobija oblik

$$\Phi = G\left(t, u, \frac{du}{dt}, \frac{d^2u}{dt^2}, \dots\right).$$

2. *Smena promenljivih u izrazima koji sadrže parcijalne izvode*
Neka je dat izraz

$$\Psi = F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}, \dots\right),$$

koji sadrži nezavisno promenljive x i y, funkciju $z = z(x,y)$ i njene parcijalne izvode proizvoljnog reda.

(A) Transformisati izraz Ψ uvođenjem novih nezavisno promenljivih u i v, koje su sledećim jednakostima:

$$u = \varphi(x,y), \quad v = \psi(x,y),$$

povezane sa nezavisno promenljivima x i y, gde su funkcije φ i ψ dovoljan broj puta diferencijabilne i $\frac{D(\varphi,\psi)}{D(x,y)} \neq 0$ u nekoj oblasti. Diferenciranjem izraza $z = z(u,v) = z[\varphi(x,y), \psi(x,y)]$ dobija se

$$(3) \qquad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial \varphi}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial \psi}{\partial x},$$

$$(4) \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial \varphi}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial \psi}{\partial y},$$

tj. dobijaju se parcijalni izvodi $\frac{\partial z}{\partial x}$ i $\frac{\partial z}{\partial y}$ u funkciji od parcijalnih izvoda $\frac{\partial z}{\partial u}$ i $\frac{\partial z}{\partial v}$, što zatim zamenjujemo u dati izraz.

Ako su nove nezavisno promenljivih u i v povezane sa starim nezavisno promenljivima x i y jednakostima

$$x = f(u,v), \quad y = g(u,v),$$

gde su funkcije f i g dovoljan broj puta diferencijabilne i $\frac{D(f,g)}{D(u,v)} \neq 0$ u nekoj oblasti, onda se diferenciranjem izraza $z = z(x,y) = z[f(u,v), g(u,v)]$ dobija sistem

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial f}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial g}{\partial v}, \tag{5}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial f}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial g}{\partial v}, \tag{6}$$

iz koga možemo izraziti parcijalne izvode $\frac{\partial z}{\partial x}$ i $\frac{\partial z}{\partial y}$ u funkciji od parcijalnih izvoda $\frac{\partial z}{\partial u}$ i $\frac{\partial z}{\partial v}$.

(B) Transformisati izraz Ψ uvođenjem novih nezavisno promenljivih u i v i nove funkcije $w = w(u,v)$, koje su sledećim jednakostima

$$u = \varphi(x,y,z), \quad v = \psi(x,y,z), \quad w = \eta(x,y,z),$$

povezane sa promenljivima x, y i z, gde su funkcije φ, ψ i η dovoljan broj puta diferencijabilne i $\frac{D(\varphi,\psi,\chi)}{D(x,y,z)} \neq 0$ u nekoj oblasti. Diferenciranjem izraza $w = w(u,v) = w[\varphi(x,y,z), \psi(x,y,z)]$ dobija se

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x}, \text{ tj.}$$

$$\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial z} \cdot \frac{\partial z}{\partial x} = \frac{\partial w}{\partial u}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial w}{\partial v}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial x}\right) \tag{7}$$

odakle možemo izraziti $\frac{\partial z}{\partial x}$ u funkciji od $\frac{\partial w}{\partial u}$ i $\frac{\partial w}{\partial v}$ i analogno

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial y}, \text{ tj.}$$

$$\frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial z} \cdot \frac{\partial z}{\partial y} = \frac{\partial w}{\partial u}\left(\frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial w}{\partial v}\left(\frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial y}\right), \tag{8}$$

odakle izražavamo $\frac{\partial z}{\partial y}$ u funkciji od $\frac{\partial w}{\partial u}$ i $\frac{\partial w}{\partial v}$.

Neka su nove nezavisno promenljive u i v i nova funkcija $w = w(u,v)$ povezane sa promenljivima x, y i z jednakostima

$$x = f(u,v,w), \quad y = g(u,v,w), \quad x = h(u,v,w),$$

gde su funkcije f, g i h dovoljan broj puta diferencijabilne i $\frac{D(f,g,h)}{D(x,y,z)} \neq 0$ u nekoj oblasti. Diferenciranjem izraza $z = z(x,y) = z[f(u,v,w), g(u,v,w)]$ dobija se sistem

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}, \text{ tj.}$$

$$\frac{\partial h}{\partial u} + \frac{\partial h}{\partial w} \cdot \frac{\partial w}{\partial u} = \frac{\partial z}{\partial x}\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial u}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial g}{\partial u} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial u}\right),$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}, \text{ tj.}$$

$$\frac{\partial h}{\partial v} + \frac{\partial h}{\partial w} \cdot \frac{\partial w}{\partial v} = \frac{\partial z}{\partial x}\left(\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial v}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial g}{\partial v} + \frac{\partial g}{\partial w}\frac{\partial w}{\partial v}\right)$$

iz koga možemo izraziti parcijalne izvode $\frac{\partial z}{\partial x}$ i $\frac{\partial z}{\partial y}$ u funkciji od $\frac{\partial w}{\partial u}$ i $\frac{\partial w}{\partial v}$.

Zadatak 3. *Transformisati jednačinu $y''' - x^3y'' + xy' - y = 0$, uzimajući za novu funkciju $u = u(t)$, pri čemu je $x = 1/t$, $y = u/t$.*

Rešenje: Imamo da je $\frac{dy}{dt} = \frac{t\frac{du}{dt} - u}{t^2}$ i $\frac{dx}{dt} = -\frac{1}{t^2}$, tj. $\frac{dt}{dx} = -t^2$, odakle je

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = u - t\frac{du}{dt},$$

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dt} \cdot \frac{dt}{dx} = \frac{d}{dt}\left(u - t\frac{du}{dt}\right) \cdot (-t^2) = t^3\frac{d^2u}{dt^2},$$

$$y''' = \frac{dy''}{dx} = \frac{dy''}{dt} \cdot \frac{dt}{dx} = -t^2 \cdot \frac{d}{dt}\left(t^3\frac{d^2u}{dt^2}\right) = -t^2\left(t^3\frac{d^3u}{dt^3} + 3t^2\frac{d^2u}{dt^2}\right).$$

Zamenom u jednačinu dobija se

$$t^5u_t''' + (3t^4 + 1)u_t'' + u_t' = 0.$$

Zadatak 4. *Transformisati jednačinu $y' \cdot y''' - 3(y'')^2 = 0$, uzimajući y za novu nezavisno promenljivu, a x za novu funkciju.*

Rešenje: Kao u prethodnom zadatku, označimo novu nezavisno promenljivu sa t, a novu funkicju sa $u = u(t)$, gde je $y = t$, $x = u$. Onda je

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{\frac{du}{dt}} = \frac{1}{u_t'},$$

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dt} \cdot \frac{dt}{dx} = \frac{d}{dt}\left(\frac{1}{u_t'}\right) \cdot \frac{1}{\frac{du}{dt}} = -\frac{u_t''}{(u_t')^2} \cdot \frac{1}{u_t'} = -\frac{u_t''}{(u_t')^3},$$

$$y''' = \frac{dy''}{dx} = \frac{d}{dt}\left(-\frac{u_t''}{(u_t')^3}\right) \cdot \frac{1}{u_t'} = -\frac{u_t'''u_t' - 3(u_t'')^2}{(u_t')^5},$$

Zamenom u zadatoj jednačini se dobija

$$\frac{3(u_t'')^2 - u_t'''u_t' - 3(u_t'')^2}{(u_t')^6} = 0 \quad \Rightarrow \quad u_t''' \cdot u_t' = 0 \ \text{ tj. } \ \frac{d^3x}{dy^3} \cdot \frac{dx}{dy} = 0\,.$$

Zadatak 5. *Transformisati jednačinu* $y\,y' + xy^2 + x^3 = 0$*, uzimajući za novu funkciju* $u = u(t)$*, pri čemu su formule smene promenljivih date sa*

(1) $$u^2 - y^2 - x^2 = 0, \quad x^2 - t^2 + u^2 = 0\,.$$

Rešenje: Diferenciranjem jednačine (1) po t dobija se

$$u\frac{du}{dt} - y\frac{dy}{dt} - x\frac{dx}{dt} = 0, \quad x\frac{dx}{dt} - t + u\frac{du}{dt} = 0,$$

odakle rešavanjem po $\frac{dx}{dt}$ i $\frac{dy}{dt}$, imamo da je

$$\frac{dx}{dt} = \frac{t - u\cdot u_t'}{x}, \quad \frac{dy}{dt} = \frac{2u\cdot u_t' - t}{y} \quad \left(u_t' = \frac{du}{dt}\right).$$

Kako je $y' = \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{x\big(2u\cdot u_t' - t\big)}{y\big(t - u\cdot u_t'\big)}$, zamenom u polaznoj jednačini dobija se

$$xu\,u_t'\big(2 - y^2 - x^2\big) - xt(1 - y^2 - x^2) = 0\,.$$

Iz formula smene promenljivih je $-y^2 - x^2 = -u^2$, pa je transformisana jednačina oblika

$$u\,u_t' = \frac{t(1-u^2)}{2-u^2}\,.$$

Zadatak 6. *Transformisati parcijalnu diferencijalnu jednačinu*

$$xy\frac{\partial^2 z}{\partial x^2} - (x^2+y^2)\frac{\partial^2 z}{\partial x\partial y} + xy\frac{\partial^2 z}{\partial y^2} + y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.$$

uzimajući za nove nezavisno promenljive $u = \dfrac{x^2+y^2}{2}$*,* $v = xy$*.*

Rešenje: Kako su u ovom slučaju nove nezavisno promenljive zadate u funkciji od starih nezavisno promenljivih, tj. $u = u(x,y) = \dfrac{x^2+y^2}{2}$, $v = v(x,y) = xy$, to je $z(u,v) = z[u(x,y), v(x,y)]$, pa je

(1) $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = x\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v},$$

(2) $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v},$$

$$(3)\qquad \frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial u} + x\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + y\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right),$$

$$(4)\qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = y\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial z}{\partial v} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right),$$

$$(5)\qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial u} + y\,\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + x\,\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right).$$

I kako je

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x} = x\,\frac{\partial^2 z}{\partial u^2} + y\,\frac{\partial^2 z}{\partial u \partial v},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x} = x\,\frac{\partial^2 z}{\partial u \partial v} + y\,\frac{\partial^2 z}{\partial v^2},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y} = y\,\frac{\partial^2 z}{\partial u^2} + x\,\frac{\partial^2 z}{\partial u \partial v},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y} = y\,\frac{\partial^2 z}{\partial u \partial v} + x\,\frac{\partial^2 z}{\partial v^2},$$

zamenom prethodnih jednakosti u (3)–(5) dobija se

$$(3')\qquad \frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial u} + x^2\frac{\partial^2 z}{\partial u^2} + 2xy\,\frac{\partial^2 z}{\partial u \partial v} + y^2\frac{\partial^2 z}{\partial v^2},$$

$$(4')\qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z}{\partial v} + xy\frac{\partial^2 z}{\partial u^2} + (x^2+y^2)\frac{\partial^2 z}{\partial u \partial v} + xy\frac{\partial^2 z}{\partial v^2},$$

$$(5')\qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial u} + y^2\frac{\partial^2 z}{\partial u^2} + 2xy\,\frac{\partial^2 z}{\partial u \partial v} + x^2\frac{\partial^2 z}{\partial v^2}.$$

Zamenom (1), (2), (3'), (4') i (5') u polaznoj jednačini, dobija se

$$0 = \left(4x^2y^2 - (x^2+y^2)^2\right)\frac{\partial^2 z}{\partial u \partial v} + 4xy\frac{\partial z}{\partial u} = \left(4v^2 - 4u^2\right)\frac{\partial^2 z}{\partial u \partial v} + 4v\frac{\partial z}{\partial u}\,.$$

Dakle, polazna PDJ datom smenom promenljivih ima oblik

$$\left(v^2 - u^2\right)\frac{\partial^2 z}{\partial u \partial v} + v\frac{\partial z}{\partial u} = 0\,.$$

Zadatak 7. *Transformisati parcijalnu diferencijalnu jednačinu*

$$x\,\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x}{z}$$

uzimajući za nove nezavisno promenljive $u = 2x - z^2$ *i* $v = \dfrac{y}{z}$.

Rešenje: Kao i u prethodnom zadatku imamo da je

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\Big(2 - 2z\frac{\partial z}{\partial x}\Big) + \frac{\partial z}{\partial v}\Big(-\frac{y}{z^2}\frac{\partial z}{\partial x}\Big),$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\Big(-2z\frac{\partial z}{\partial y}\Big) + \frac{\partial z}{\partial v}\Big(\frac{1}{z} - \frac{y}{z^2}\frac{\partial z}{\partial y}\Big),$$

odakle je

$$\frac{\partial z}{\partial x} = \frac{2\dfrac{\partial z}{\partial u}}{1 + 2z\dfrac{\partial z}{\partial u} + \dfrac{y}{z^2}\dfrac{\partial z}{\partial v}}, \qquad \frac{\partial z}{\partial y} = \frac{\dfrac{1}{z}\dfrac{\partial z}{\partial v}}{1 + 2z\dfrac{\partial z}{\partial u} + \dfrac{y}{z^2}\dfrac{\partial z}{\partial v}}.$$

Tada data jednačina dobija oblik

$$\frac{2x\dfrac{\partial z}{\partial u} + \dfrac{y}{z}\dfrac{\partial z}{\partial v}}{1 + 2z\dfrac{\partial z}{\partial u} + \dfrac{y}{z^2}\dfrac{\partial z}{\partial v}} = \frac{x}{z}.$$

Uzevši u obzir da je $2x = u + z^2$, $\dfrac{y}{z} = v$, $\dfrac{x}{z} = \dfrac{u + z^2}{2z}$, dobija se

$$\frac{\partial z}{\partial v} = \frac{z(u + z^2)}{v(z^2 - u)}, \quad z^2 \neq u.$$

8. *Dokazati da PDJ* $\Big(\dfrac{\partial z}{\partial x}\Big)^2 + \Big(\dfrac{\partial z}{\partial y}\Big)^2 = 0$ *ne menja oblik uzimajući za nove nezavisno promenljive* u *i* v, *ako je* $x = uv$, $y = \dfrac{u^2 - v^2}{2}$.

Rešenje: U ovom slučaju stare nezavisno promenljive x i y zadate su u funkciji od novih nezavisno promenljivih, tj. $x = x(u,v) = uv$, $y = y(u,v) = \dfrac{u^2 - v^2}{2}$. Diferenciranjem $z = z(x,y) = z[\,x(u,v), y(u,v)\,]$ kao složene funkcije, dobija se sistem

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial x} = v\,\frac{\partial z}{\partial x} + u\,\frac{\partial z}{\partial y},$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial x} = u\,\frac{\partial z}{\partial x} - v\,\frac{\partial z}{\partial y},$$

čijim rešavanjem po $\dfrac{\partial z}{\partial x}$ i $\dfrac{\partial z}{\partial y}$, dobijamo

$$\frac{\partial z}{\partial x} = \frac{v\dfrac{\partial z}{\partial u} + u\dfrac{\partial z}{\partial v}}{u^2 + v^2}, \qquad \frac{\partial z}{\partial y} = \frac{u\dfrac{\partial z}{\partial u} - v\dfrac{\partial z}{\partial v}}{u^2 + v^2}, \quad u^2 + v^2 \neq 0.$$

Dakle,

$$0 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{\left(v\dfrac{\partial z}{\partial u} + u\dfrac{\partial z}{\partial v}\right)^2 + \left(u\dfrac{\partial z}{\partial u} - v\dfrac{\partial z}{\partial v}\right)^2}{(u^2+v^2)^2}$$

$$= \frac{\left(\dfrac{\partial z}{\partial u}\right)^2 + \left(\dfrac{\partial z}{\partial v}\right)^2}{(u^2+v^2)^2},$$

odnosno dobija se PDJ oblika $\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = 0$.

Zadatak 9. *Transformisati parcijalnu diferencijalnu jednačinu*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = z$$

uzimajući za nove nezavisno promenljive $u = \dfrac{x+y}{2}$, $v = \dfrac{x-y}{2}$ *i za novu funkciju* $w = z\,e^y$.

Rešenje: U ovom slučaju je $w(u,v) = w[u(x,y), v(x,y)]$, pa je

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} \Rightarrow e^y\frac{\partial z}{\partial x} = \frac{\partial w}{\partial u}\cdot\frac{1}{2} + \frac{\partial w}{\partial v}\cdot\frac{1}{2}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{e^{-y}}{2}\left(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\right)$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} \Rightarrow z\,e^y + e^y\frac{\partial z}{\partial y} = \frac{\partial w}{\partial u}\cdot\frac{1}{2} + \frac{\partial w}{\partial v}\cdot(-\frac{1}{2})$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{e^{-y}}{2}\left(\frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}\right) - z$$

Odavde je

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{e^{-y}}{2}\Big(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\Big)\right) =$$

$$= \frac{e^{-y}}{2}\left[\frac{\partial}{\partial u}\Big(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\Big)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\Big(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\Big)\frac{\partial v}{\partial x}\right] =$$

$$= \frac{e^{-y}}{2}\left[\frac{1}{2}\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial u \partial v} + \frac{1}{2}\frac{\partial^2 w}{\partial v^2}\right],$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{e^{-y}}{2}\Big(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\Big)\right) = -\frac{e^{-y}}{2}\Big(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\Big) +$$

$$+ \frac{e^{-y}}{2}\left[\frac{\partial}{\partial u}\Big(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\Big)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\Big(\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\Big)\frac{\partial v}{\partial y}\right] =$$

$$= \frac{e^{-y}}{4}\Big(-2\frac{\partial w}{\partial u} - 2\frac{\partial w}{\partial v} + \frac{\partial^2 w}{\partial u^2} - \frac{\partial^2 w}{\partial v^2}\Big).$$

Zamenom u zadatoj jednačini dobija se

$$\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial u \partial v} - 2\frac{\partial w}{\partial v} = 4w\,.$$

Zadatak 10. *Transformisati parcijalnu diferencijalnu jednačinu*

$$(x-z)\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

uzimajući x za novu funkciju, a y i z za nove nezavisno promenljive.

Rešenje: Stavimo da je $w = x$, $u = y$ $v = z$, $w = w(u, v)$. Kao i u prethodnom zadatku imamo da je

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} \ \Rightarrow\ 1 = \frac{\partial x}{\partial y}\cdot 0 + \frac{\partial x}{\partial z}\cdot\frac{\partial z}{\partial x} \ \Rightarrow\ \frac{\partial z}{\partial x} = \frac{1}{\dfrac{\partial x}{\partial z}},$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} \ \Rightarrow\ 0 = \frac{\partial x}{\partial y}\cdot 1 + \frac{\partial x}{\partial z}\cdot\frac{\partial z}{\partial y} \ \Rightarrow\ \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial x}{\partial y}}{\dfrac{\partial x}{\partial z}}\,.$$

Dakle, biće

$$(x-z)\frac{1}{\dfrac{\partial x}{\partial z}} - y\,\frac{\dfrac{\partial x}{\partial y}}{\dfrac{\partial x}{\partial z}} = 0 \ \Rightarrow\ \frac{\partial x}{\partial y} = \frac{x-z}{y}\,.$$

Zadatak 11. *Transformisati parcijalnu diferencijalnu jednačinu*

$$(y-z)\frac{\partial z}{\partial x} + (y+z)\frac{\partial z}{\partial y} = 0$$

uzimajući x za novu funkciju, a $u = y - z$ i $v = y + z$ za nove nezavisno promenljive.

Rešenje: Stavimo da je $w = x$, tako da je $w = w(u, v) = w[u(y, z), v(y, z)]$. Onda je

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} \ \Rightarrow\ 1 = \frac{\partial x}{\partial u}\Big(-\frac{\partial z}{\partial x}\Big) + \frac{\partial x}{\partial v}\cdot\frac{\partial z}{\partial x} \ \Rightarrow\ \frac{\partial z}{\partial x} = \frac{1}{\dfrac{\partial x}{\partial v} - \dfrac{\partial x}{\partial u}}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} \ \Rightarrow\ 0 = \frac{\partial x}{\partial u}\Big(1 - \frac{\partial z}{\partial y}\Big) + \frac{\partial x}{\partial v}\Big(1 + \frac{\partial z}{\partial y}\Big)$$

$$\Rightarrow\ \frac{\partial z}{\partial y} = \frac{\dfrac{\partial x}{\partial u} + \dfrac{\partial x}{\partial v}}{\dfrac{\partial x}{\partial u} - \dfrac{\partial x}{\partial v}}\,.$$

Data jednačina sada ima oblik

$$u\,\frac{1}{\dfrac{\partial x}{\partial v}-\dfrac{\partial x}{\partial u}}+v\,\frac{\dfrac{\partial x}{\partial u}+\dfrac{\partial x}{\partial v}}{\dfrac{\partial x}{\partial u}-\dfrac{\partial x}{\partial v}}=0\;\Rightarrow\;\frac{\partial x}{\partial u}+\frac{\partial x}{\partial v}=\frac{u}{v}\,.$$

Zadatak 12. *Transformisati parcijalnu diferencijalnu jednačinu*

$$\left(z\frac{\partial z}{\partial x}\right)^2+\left(y\frac{\partial z}{\partial y}\right)^2=z^2\frac{\partial z}{\partial x}\,\frac{\partial z}{\partial y},$$

uzimajući za novu funkciju $w=w(u,v)$, *ako je* $x=u\,e^w$, $y=v\,e^w$, $z=w\,e^w$.

Rešenje: Nove nezavisno promenljive u, v i nova funkcija w zadate su u funkciji od starih nezavisno promenljivih x, y i funkcije z. Zato je $z=z(x,y)=z[\,x(u,v,w),y(u,v,w)\,]$, odakle se dobija sistem

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u},\ \text{tj.}$$

$$\text{(a)}\qquad \frac{\partial w}{\partial u}\,e^w+w\,e^w\,\frac{\partial w}{\partial u}=\frac{\partial z}{\partial x}\left(e^w+u\,e^w\,\frac{\partial w}{\partial u}\right)+\frac{\partial z}{\partial y}\,v\,e^w\,\frac{\partial w}{\partial u},$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v},\ \text{tj.}$$

$$\text{(b)}\qquad \frac{\partial w}{\partial v}\,e^w+w\,e^w\,\frac{\partial w}{\partial v}=\frac{\partial z}{\partial x}\,u\,e^w\,\frac{\partial w}{\partial u}+\frac{\partial z}{\partial y}\left(e^w+v\,e^w\,\frac{\partial w}{\partial v}\right)\,.$$

Iz (a) i (b) se dobija

$$\frac{\partial z}{\partial x}=\frac{(1+w)\dfrac{\partial w}{\partial u}}{1+u\,\dfrac{\partial w}{\partial u}+v\,\dfrac{\partial w}{\partial v}},\qquad \frac{\partial z}{\partial y}=\frac{(1+w)\dfrac{\partial w}{\partial v}}{1+u\,\dfrac{\partial w}{\partial u}+v\,\dfrac{\partial w}{\partial v}}\,,$$

što zamenom u jednačinu daje

$$\left(u\frac{\partial w}{\partial u}\right)^2+\left(v\frac{\partial w}{\partial v}\right)^2=w^2\frac{\partial w}{\partial u}\,\frac{\partial w}{\partial v}\,.$$