Monograph of Academy of Nonlinear Sciences
Advances in Nonlinear Sciences II - Sciences and Applications, Belgrade
Non consonance in theory of Mechanics: Knowledge and Science, 2008, Vol. 2, pp. 160-2222.

NON CONSONANCE IN THE THEORY OF MECHANICS: KNOWLEDGE AND SCIENCE

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Instead of preface. In the preface of the book [1] it was understood that mechanics is an exact natural science; it is as exact as mathematics or, more precisely, it is more exact it if its assertions need not only mathematical proofs, bat also verification by nature. From the science's standpoint, it can be concluded that terms "science" and "knowledge" are not the same. The sufficient proof for such statement is many experts cannot agree over many issues in partial and theoretical mechanics. Science is not made only of present knowledge, but also of checking and upgrading of that knowledge; it raises a suspicion toward the present knowledge, and it seeks for the new one in the wide theory, starting from its standpoint. The very standpoint is raising the different opinions in mathematic natural sciences. For instance, there are two explanations of the second axiom of Newton's mechanics, which is resulting in big mistakes in practical mechanics. The result of using of non-standardized mathematic knowledge and its limitations by theoretical physicians, is the description of the world that does not exist.

Keywords: Knowledge, science, essence; classical and celestial Mechanics.

1. ESENCIAL PROBLEMS OF MATHEMATICAL MECHANICS

We see mathematics as the absolute truth; however at the higher level of knowledge it is changeable, many things are added, and constructed according to the check ups and needs in practice. We can find definitions of a vector, as the elements of set of vectors, or as the elements of vectors' space, however, none of these are defining vectors in the correct way.This is also observed at the other levels of knowledge. Widely spread incomplete definitions, on important issues in physics are present, such as: "The quantity of motion of the system of a material points is equal to the vectors' sum $\mathbf{K} = \sum_i m_i \mathbf{v}_i$," although it is not correlated to the sum of connected vectors, like with vectors of velocities $\mathbf{v}_i$.

It is a fact that many mathematical experts do not have the same opinion and do not agree over mathematical issues. It is natural, since mathematics was developing as the tree: its branches are growing, they can be modelled, cut etc. Mathematics is not just a calculation, the way it is understood at the elementary level, it is wisdom, and derive of the human mind...Thus, mathematics is the key issue in the theory of mechanics.

For better understanding of the basic problems of science of the motion of a body, it is necessary to separate the important mathematic terms, that are used in mechanics, and on which there are no unified consonant.

Numbers are inevitable in any mathematical field. Let us denote by a letter $\mathcal{R}$, all rational numbers. As Mechanics is not fully the rational theory, with the characteristics : *mass* m, $dim\, m = M$, *length* l, $dim\, l = L$ and *time* t, $dim\, t = T$; or in the other words: $m \in R(M)$, $l \in R(L)$, $t \in R(T)$, or: mass m has a dimension M, length l has a L, and time t has T. The different dimensions cannot be summarized, nor equalized because

$$M \neq L \neq T. \tag{1.1}$$

Denote by $R(M)$ the set of all of real numbers, the set of all real numbers of length with $R(L)$ and the set of the real numbers of time with $R(T)$. These are sets of denominate numbers of different essentials. Let's repeat the known fact that the elements of these sets cannot be summarized, but they can be multiplied

$$m^a \cdot l^b \cdot t^c; \quad (a, b, c) \in \mathcal{R}.$$

These products are now new sets of denominated numbers $R(MLT)$, with existing physical components, thus they are preserving the essence of mechanics. All other values and relations of mechanics are calculated by using the mentioned three attributes. If different, according to the preprinciples of existence [1] the would not belong to the theory of motion of body. Besides the scalar values, determined by the denominate numbers, mechanics has other characteristics: velocities of material points, angular impulses of motion, acceleration, force, moment of forces and moment of impulses of motion. Such values are often shortly described by vectors and tensors.

Vectors. Vector is a mathematical term that has the numerical value pointed at the certain direction. Vector can be written in the form

$$\mathbf{v} = v\mathbf{v}_0, \quad \mathbf{v}_0 = \frac{\mathbf{v}}{v},$$

where v is a numeric value (size, quantity, scalar) of vector $\mathbf{v}$, and $\mathbf{v}_0$ is the unit, oriented, non-dimensional vector, so it is

$$dim\, \mathbf{v} = dim\, v.$$

In geometry the directed segment $\boldsymbol{\rho}$ is the typical example of vector from the set of vectors $\mathbf{V}(L)$, whose length is $\rho \in R(L)$, and $\boldsymbol{\rho}_0 = \frac{\boldsymbol{\rho}}{\rho}$ is the orientated non-dimensional unit vector.

As the length of l, $l \in R(L)$ or ρ between the two points can be calculated by using three coordinate numbers l_1, l_2, l_3 from $R(L)$, also the vector $\boldsymbol{\rho}$ can be written by using these numbers and three independently oriented vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \in \mathbf{V}_3$, as

$$\boldsymbol{\rho} = \sum_{i=1}^{3} \rho^i \mathbf{e}_i; \quad |\, \mathbf{e}_i \,| = 1, \quad \mathbf{e}_i \perp \mathbf{e}_j .$$

Analogous to the form $\rho \in R(L)$, it is possible to write $\boldsymbol{\rho} \in \mathbf{V}_3$, or $\rho^i \in R^3(L)$, where $\mathbf{V}_3$ is the defined set of three vectors, and ρ^i is the set of real numbers $\rho^i \in R(L)$. Accordingly to the above mentioned, we have: vectors set R^n on linear base of vector $\mathbf{V}_n$. The mentioned approach to the term of vector is not accidental, as some of the generalized "vectors' space" and its use leads to the different opinions in mechanics.

Addition of vectors. In vectors' calculus and its application in mechanics, three vectors are present: *free, slide and constrained vectors.* Free vector is determined by size, imaginary direction and orientation. (Fig.1a); slide vector is the one whose direction is determined by a line coinciding with axis of vector (Fig. 1b). Constrained vector for point is defined with the corresponding point, by size and oriented direction (Fig. 1c). The differences between them are considerable.

Figure 2 is showing the obvious difference between vectors.

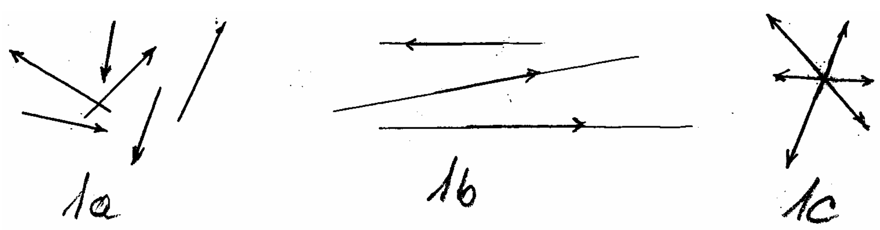

Figure 1. Vectors

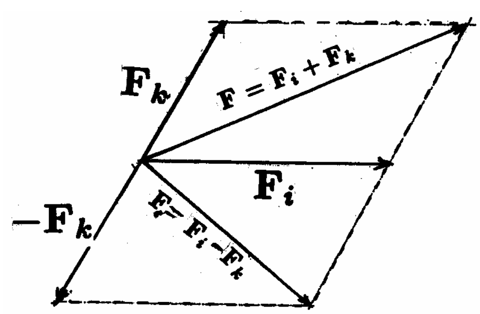

Figure 2. Radius vectors

Positions of points M_1 and M_2 can be determined by the variety of radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, whose poles P_i are the arbitrary points of the observed plane line. However vector $\boldsymbol{\rho} = \rho\boldsymbol{\rho}_o$ is the only one

$$\mathbf{r}_2 - \mathbf{r}_1 = \ldots = \mathbf{r}_\nu - \mathbf{r}_{\nu-1} = \boldsymbol{\rho}, \tag{1.2}$$

if points P_i, M_2 and M_1 belong to the same plain. If the vector $\boldsymbol{\rho}$ belongs to straight it is called *slide vector* or *vector constrained to live.* Forces in mechanics are described by such vectors. Some authors even described the force with "force is a vector". Slide vectors in the plane, can be reduced to the intersection point.

The first rule of addition of vectors is defined by the rule of parallelogram: *the sum of two vectors is equal to the diagonal of the parallelogram formed bay these two vectors.*(Fig. 3).

Reduction of the system of the slide vectors to the point in the compact domain (body), is presented in the figure (Fig. 4). Let $\mathbf{F}_i$ be vector an the M_i. The state of vector $\mathbf{F}_i$ in the point M_1 will be not changed if we add two other forces $\mathbf{F}_k$ and $-\mathbf{F}_k$ such that $\mathbf{F}_k + (-\mathbf{F}_k) = \mathbf{0}$. Thus at the point M_k we obtain vector $\mathbf{F} = \mathbf{F}_i + \mathbf{F}_k$, and vector of moment of force $\boldsymbol{M}_k = \boldsymbol{\rho}_i \times \mathbf{F}_i$.

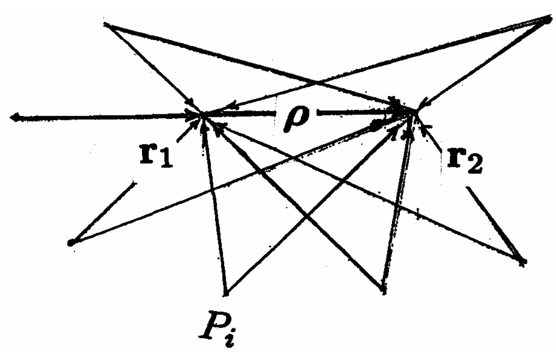

Figure 3. Vector sum

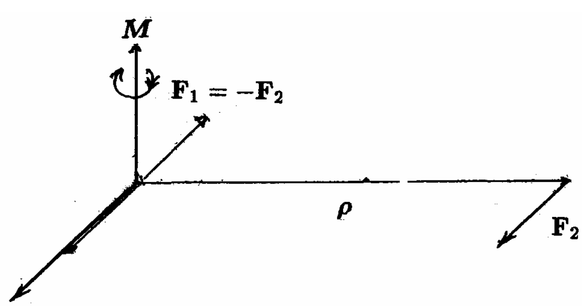

Figure 4. Reduction on vector to another point

Using the same approach, all forces can be reduced at a point C, to one *main vector*

$$\mathbf{F}_C = \sum_i \mathbf{F}_i, \qquad dim\,\mathbf{F}_i = MLT^{-2},$$

and one vector of different dimension - *main moment*

$$\mathbf{M}_C = \sum_i \boldsymbol{\rho}_i \times \mathbf{F}_i; \quad dim\,\mathbf{M} = ML^2T^{-1}. \tag{1.3}$$

From the above considerations we obtain the following theorem: *Every system of vectors $\mathbf{F}_i$, freely positioned in a compact domain, can be reduced to the main vector $\mathbf{F}_C$ at an arbitrary point C, and to the main moment $\mathbf{M}_C$, which is equal to the sum of vector products radius of vectors $\boldsymbol{\rho}_i$ and $\mathbf{F}_i$.*

Note that the main vector is not the resultant of the system of all vectors, reduced to the point C, but together with the main moment $\mathbf{M}_C$ it is equivalent to the system of all vectors in the existing homogenous configuration.

The use of algebraic operations on constrained vectors, as well as on free vectors leads to the inconsistent results.

Example. The velocity of material point is a typical constrained vector connected to the material point. Suppose two material points M_1 and M_2 are two vehicles, running at the strait road parallel to each other, with the same velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. The reader should try to sum them ?!

The sum of two constrained vectors for non-congruent points is not corresponding to the definition of sum of two free vectors, and even less to the driving practice of observed vehicles. In order to avoid the mentioned problem of complexity of vectors' calculus in summing different vectors, we use the scalar calculus in analytical mechanics and in differential geometry.

Multidimensional vectors "spaces". Opening the quotation marks, the author would like to emphasis the difference of the term space as the natural reality, from the various mathematical "space". In Euclidean geometry, the position of ani point with refer to the observation point can be determined using the vector of position: this means that we have to know three data - size of the vector of position, direction and orientation, or three defined components of the vector

$$\mathbf{r} = \mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3 = \sum_{i=1}^{3} y^i \mathbf{e}_i := y^i \mathbf{e}_i,$$

where $\mathbf{e}_i \in \mathbf{E}_3$ are three orthonormal unit vectors, directed to each other. Contrary to the term component $\mathbf{y}_i \in \mathbf{E}_3$, numbers or scalar function $y^i \in R^3$ is called vector's coordinate or "vectors", where the vector's base is known in advance. Number of basic vectors and number of vector's coordinate are in most cases equal, however it is not a rule,especially in Euclidean geometry. Thus, if there is N mutually non constrained points, their positions are determined by N vectors $\mathbf{r}_\nu$; $,\nu = 1,\ldots,N$:

$$\mathbf{r}_\nu = \sum_{i=1}^{3} y_\nu^i \mathbf{e}_i,$$

where $3N$ coordinates y_ν^i are appearing over the base $\mathbf{E}_3$. Here we should not forget that it is not the $3N$-coordinate space, it is in fact three dimensional vector in $\mathbf{E}_3$.

$$\sum_{\nu=1}^{N} \mathbf{r}_\nu = (\sum_{\nu=1}^{N} y_\nu^1)\mathbf{e}_1 + (\sum_{\nu=1}^{N} y_\nu^2)\mathbf{e}_2 + (\sum_{\nu=1}^{N} y_\nu^3)\mathbf{e}_3. \tag{1.4}$$

Similarli with refer to some other vector base $\mathbf{g}_i$; $i = 1,2,3$ can be written $\boldsymbol{\rho} = f^i \mathbf{g}_i \in \mathbf{R}_3$, or $(f^1, f^2, f^3) \in R^3$. If, for example, $y^3 = 0$, or $f^3 = 0$, we will have two dimensional coordinate vectors $(y^1, y^2) \in E^2$, or in general R^2, where the relation $f^3 = 0$ is present.

Let's notify that vectors from $\mathbf{R}^*$ i $\mathbf{R}$ are linear ones at the base vectors, regardless of non-linear level of it is coordinate functions f^i. That's way we call them linear (read: vector's) space. If the linear level is measured with refer to the level of coordinate function f^i, then they are nonlinear vector space. In the curvilinear coordinate systems of x^1, x^2, x^3, which are in mutual transformation with linear c00rdinate vector y^i, e.g. $y^i = y^i(x^1, x^2, x^3)$, for which is ;

$$\mid \frac{\partial y^i}{\partial x^j} \mid \neq 0,$$

the differential of vector can be written in the form:

$$d\boldsymbol{\rho} = \frac{\partial \mathbf{r}}{\partial y^i} dy^i = \frac{\partial \mathbf{r}}{\partial x^i} dx^i = dy^i \mathbf{e}_i = \frac{\partial yi}{\partial x^j} dx^j \mathbf{e}_i = \mathbf{g}_j dx^j, \tag{1.5}$$

where

$$\mathbf{g}_j(x) = \frac{\partial y^i}{\partial x^j} \mathbf{e}_i. \tag{1.6}$$

Thus, metric in $d\rho^2 \in E^3$ has the form

$$d\rho^2 = \frac{\partial \mathbf{r}}{\partial y^i} \cdot \frac{\partial \mathbf{r}}{\partial y^j} dy^i dy^j = \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j} dx^i dx^j = \delta_{ij} dy^i dy^j = g_{ij}(x) dx^i dx^j, \tag{1.7}$$

where

$$g_{ij}(x^1, x^2, x^3) \in E^3 \subset R(L); \quad i,j = 1,2,3$$

is the metric tensor of Euclidean geometry.

Multicoordinated manifolds of differential geometry. The term "Multidimensional" or "Multicoordinated" stands for geometrical forms, which are described by one coordinate or component. The most simple one and thus the most general manifold is N points M_ν, with the positions determined by N vector position (1.4). Number $3N$ of coordinates y^i_ν is found in the expression $y_\nu \in E^{3N} \subset R(L)$, but one should makes the difference between E^{3N} and $\mathbf{E}_3$, because, all the mentioned vectors $\mathbf{r}_\nu$ can be summarize in a three dimensional vector (1.4). This is possible because all the vectors have a chosen starting point. However, if while observing vectors O. $\Delta\mathbf{r}_\nu$, that are individually constrained for the own ν-th points, we are obtaining N coordinate system, where the poles are exactly the points M_ν.

It is not possible to summarize these constrained vectors $\Delta\mathbf{r}_\nu$ without deducting them to the point, which is again not possible without parallel moves, or adding of vectors of moment (curl) of this vectors. The coherence between $3N$ coordinates of vectors $\Delta\mathbf{r}_\nu$ and $3N$ coordinates of vectors $\mathbf{e}_{\nu 1}, \mathbf{e}_{\nu 2}, \mathbf{e}_{\nu 3}$ is now there..

Through analysis of the curvilinear systems of coordinates, it is emphasized that all curvilinear coordinates x do not have dimension of length. Further, basic or coordinate vectors are not constant vectors, they are in fact appearing as the vector's function of curvilinear coordinates. It is important here to notice that vectors $\Delta\mathbf{r}_\nu$, as well as differentials $d\mathbf{r}_\nu$, are significantly different form the vectors $\mathbf{r}_\nu$, as each vector $d\mathbf{r}_\nu$ is constrained to its relevant point M_ν. Thus the vectors' summarizing is very difficult. Also, this is a reason why the differential geometry and analytical mechanics is using more scalar relations and invariants. Exchanging the linear coordinates y^i with *curvilinear coordinates* x, with general relations $y^i = y^i(x^1, x^2, x^3)$ the geometrical essence of differential $\Delta\mathbf{r}$, must not be changed, and this is provided by relation.

$$\Delta\mathbf{r} = \frac{\partial\mathbf{r}}{\partial y^i}\Delta y^i = \frac{\partial\mathbf{r}}{\partial\, y^i}\frac{\partial\, y^i}{\partial\, x^j}\Delta x^j = \Delta y^i\mathbf{e}_i = \Delta x^j\mathbf{g}_j, \tag{1.8}$$

or

$$\frac{\Delta\mathbf{r}}{\Delta s} = \frac{\partial\mathbf{r}}{\partial y^i}\frac{\partial\, y^i}{\partial x^j}\frac{\partial\, x^j}{\Delta s} = \frac{\Delta y^i}{\Delta s}\mathbf{e}_i = \frac{\Delta x^j}{\Delta s}\Delta x^j\mathbf{g}_j.$$

From here, it is shown that coordinate vectors

$$\mathbf{g}_j(x) = \frac{\partial\,\mathbf{r}}{\partial\, x^j}$$

are present as vectors' functions of curvilinear coordinates of the starting point of vector $\Delta\mathbf{r}$. The introduction of curvilinear coordinates for each point or partly for some of the points, is justified only for points, that can be written in the form *constraints*,

$$f_\mu(\mathbf{r}_1, \ldots, \mathbf{r}_N) = 0,$$

or related to linear system coordinates

$$f_\mu(y^1, \ldots, y^{3N}) = 0,$$

or equivalently in the curvilinear system of coordinates

$$f_\mu(x^1, \ldots, x^{3N}) = 0. \tag{1.9}$$

As **example**, if a point belongs to the spherical surface, described using the Descartes's coordinates, as the equation $y_1^2+y_2^2+y_3^2=c^2$, and in spherical space of coordinates ρ,ϕ,θ, with the simplified equation $\rho=c$, normally the spherical coordinates will be used.

Each of the vectors

$$d\mathbf{r}_\nu=\frac{\partial \mathbf{r}_\nu}{\partial x_\nu^i}dx_\nu^i$$

can be scalar multiplied with the relevant vectors $\frac{\partial \mathbf{r}_\nu}{\partial x_\nu^j}$, thus, being scalars, they can summarize:

$$\sum_{\nu=1}^N d\mathbf{r}_\nu\cdot\frac{\partial \mathbf{r}_\nu}{\partial x_\nu^j}=\sum_\nu^N\frac{\partial \mathbf{r}_\nu}{\partial x_\nu^i}dx_\nu^i\cdot\frac{\partial \mathbf{r}_\nu}{\partial x_\nu^j}:=\sum_\nu^N g_\nu(x)_{ij}dx_\nu^i=g_{kl}(x^1,\dots,x^{3N})dx^l,$$

where the formal equivalent x^κ is introduced for the $x^{3\kappa-2}\equiv x^{3\kappa-1}\equiv x^{3\kappa}$.

Also,

$$ds^2:=\sum_{\nu=1}^N d\mathbf{r}_\nu\cdot d\mathbf{r}_\nu=\sum_\nu^N=g_{(\nu)ij}dx_\nu^i:=g_{kl}(x^1,\dots,x^{3N})dx^l.$$

can be written. On the tensor

$$g_{ij}=g_{ji}(x^1,\dots,x^{3N}), \tag{1.10}$$

likewise an in (1.7), the metric can be established.

$$dr^2=g_{kl}(x)dx^kdx^l. \tag{1.11}$$

Accordingly to the metric tensor (1.10), one can say it is "metrics of the $3N$-dimensional space", however it is omitted to notify that such constructions is obtained through different exchanges."Multidimensional space", obtained in such a way, can have the fictive sense geometry, and not the real one. Multidimensional space of geometry is a mathematical term, which presents a number of coordinates or coordinates' manifold, used to define a position of the **system** of N points, where the systems constrains, are like in (1.9).

For the system of points linked by the final constrains (1.9), where it is understood that functions f_μ are indefinite in the possible space, e.g. in the space where the conditions are satisfied:

$$df_\mu=\frac{\partial f_\mu}{\partial x^i}dx^i=0\quad |\,\partial f_\mu\partial x^i\,|\neq 0;\quad i=1,\dots,3N,$$

it is possible to define coordinates x^i in the function $3N-k=n$ of independent generalized coordinates $q^1,\dots,q^n$, on which the metrics is done

$$ds^2=g_{\alpha\beta}dq^\alpha dq^\beta. \tag{12}$$

Tensor

$$g_{\alpha\beta}(q^1, \ldots, q^n) \in M^n \tag{13}$$

is often called the metrics tensor of n-dimensional configuration space. There is nothing unclear: at the plain school globe of our planet. many places, marked by point are defined using two independent coordinates $q^1 = \varphi$, $q^2 = \theta \in M^2$. However this the plain globe, where it is visible that the Earth is not a mathematical sphere $R = c = conct.$, but the real body. where many hills, mountains, and other places, are defined by a height.

2. GEOMETRIZATION OF MECHANICS OR DYNAMIZATION GEOMETRY

In the introductory relation it is emphasized that besides the set of number in geometry $R(L)$, the essence of mechanics is made also with $R(M)$ and $R(T)$. It is done in a simple way, by multiplying the numbers of geometrical origin with scalar values of masses $m \in R(M)$ and time $t \in R(t)$. In the introduction of its important paper, Newton wrote on May 08, 1686: "Geometry is base on mechanical practise and it is nothing more then the other part of the general mechanics, in which the skill of precise measuring is presented and proven."[2].That statement of the great scientist has the same importance even today.

The expression (1.8) is divided with the small interval of time Δt, e.g. limiting value of the relation of distance and interval of time Δt, for which the material point is moved from one position $\mathbf{r}(t)$ in the nearby other position $\mathbf{r}(t + \Delta t)$ on time is according to the definition the velocity of motion of the material point

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial y^i}\frac{\partial \mathbf{r}}{\partial x^i} = \dot{y}^i \mathbf{e}_i = \dot{x}^i \mathbf{g}_i. \tag{2.1}$$

Considering as important, it is written,

$$dim\,\mathbf{v} = dim\,\dot{y}^i = LT^{-1}. \tag{2.2}$$

and, as well as, that the velocity $\mathbf{v}_\nu$ is the constrained vector for ν point.

The product of mass m of the material point and its velocity vector $\mathbf{v}$ is called the motion impulse of material point,

$$\mathbf{p} = m\mathbf{v}. \tag{2.3}$$

Thus the impulse of material points is a vector connected to the point. The basic physical dimensions of the important term are all essences of mechanics, e.g.

$$dim\,\mathbf{p} = MLT^{-1} \tag{2.4}$$

Obviously vector $\mathbf{p}_\nu$ of material point has the same vector characteristic as the velocity vector $\mathbf{v}_n u$ in the addition process. It means that the vectors of impulses p of the mutually connected points of the system are not added as free vectors.

Sum of impulses of motion of several material points has a big importance in the system of material points. The term "system" means that the material points are connected with some constraints $f(\mathbf{r}_\nu, \mathbf{v}_\nu) = 0$, which must be taken into consideration at the addition of velocity vectors.

Accordingly, impulse vector of the ν-th material point of mass m_ν of the observed system can also be represented by the formula

$$\mathbf{p}_\nu = m_\nu \mathbf{v}_\nu = m \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \dot{q}^\alpha. \tag{2.5}$$

Scalar multiplication by coordinate vectors $\dfrac{\partial \mathbf{r}_\nu}{\partial q^\beta}$ gives vector $\mathbf{p}_\nu$ projection upon the tangential direction of q^β coordinate of the ν-th material point. We will denote it by a two-indices letter:

$$p_{\nu\beta} = m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta} \dot{q}^\alpha.$$

This is in accordance with the formula for impulse's coordinates of one material point. Regarding the fact that $p_{\nu\beta}$ impulses are scalars, it is possible to sum them up:

$$p_\beta := \sum_{\nu=1}^{N} p_{\nu\beta} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta} \dot{q}^\alpha = a_{\alpha\beta} \dot{q}^\alpha, \tag{2.6}$$

from which it can be seen that $a_{\alpha\beta}$ is an inertia tensor of the whole system:

$$a_{\alpha\beta} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta} = a_{\alpha\beta}\left(m_1, \ldots, m_N; q^0, q^1, \ldots, q^n\right). \tag{2.7}$$

A.B. Gohman [30], in his booklet, emphasizing the importance of dependence of tensor $a_{\alpha\beta}$ of the mass, for the proper interpretation of mechanics of systems with variable masses, reference to the papers of V.A. Vujičić.

By comparing the geometrical formulas and the mentioned mechanical formulas, the great similarity is observed, and for some mathematic interpretations, they are even identical. However, the clear differentiation exist:

1. we discussed the geometry using the numbers from $S(L)$, and mechanics using the numbers from the set $S(MLT)$;
2. tangent vector $\frac{d\mathbf{r}}{ds}$ in geometry is corresponds to the vector $\frac{\partial \mathbf{r}}{\partial s}\frac{ds}{dt}$ in mechanics;
3. in geometry the impulse of motion of material point, of dimension MLT, does not existence, neither do the very material points.
4. Points in geometry are not-dimensional, and material points has the dimension of mass. We can say that the material point is geometrical point, where the masse added, however big different existence, as one geometrical point can be the very same one for the indefinite set of material unequal points.

5. For physicists and natural philosophers, the difference between metric tensor (1.13) and the related inertial tensor (2.7) is important.

6. In geometry the verbs "moving, pushing, caring, ..." are used, but those words are requiring the use of time $t, dim\, t \in R(T)$. The term motion is the part of kinematic, which is a part of mechanics. Thus the correct of subtitle where the geometrical mechanics is emphasized should be repeated. The geometric Mechanics is made of system of material points of the image of multidimensional space, being only fictions and incorrect images on unreachable objects, especially in celestial mechanics.

Systems with Variable Constraints. In the case that finite constraints

$$f_\mu(x^1, \ldots, x^{3N}, t) = 0, \tag{2.8}$$

depend not only upon $x^1, \ldots, x^{3N}$ coordinates, but also explicitly on time as well, velocity conditions and those of acceleration considerably change, since the number of addends is increasing under these conditions as is obvious in the following velocity conditions:

$$\dot{f}_\mu = \frac{\partial f_\mu}{\partial y^i}\dot{y}^i + \frac{\partial f_\mu}{\partial t} = grad_\nu f_\mu \cdot \mathbf{v}_\nu + \frac{\partial f_\mu}{\partial t} = 0. \tag{2.9}$$

The variable constraints in time must satisfy the dimension equation, that is, they have to be dimensionally homogeneous. In order to achieve this homogeneity between y coordinates of L dimension and time t of dimension T, it is necessary to connect these values by some parameter κ of the dimensions L and T. Therefore, time in mechanical constraints appears in the structure of the functions containing dimension parameters, so that variable or moveable constraints, in accordance with definition (2.8) are written in the form:

$$f_\mu(y, \tau) = 0 \quad (\mu = 1, \ldots, k), \tag{2.10}$$

where τ is some real time function with definite real coefficients having physical dimensions. For the sake of brevity, instead of function τ with definite coefficients, let's introduce an additional coordinate y^0, so that it satisfies the condition

$$f_0 = y^0(\kappa, t) - \tau(t) = 0. \tag{2.11}$$

In general with y^0 coordinate, rheonomic constraints can be written in the form

$$f_\mu(y, y^0)) = f_\mu(y^0, \underbrace{y^1, \ldots, y^{3N}}_{y}) = 0, \tag{2.12}$$

while the velocity and acceleration conditions in the form (2.10) and (2.12), that is,

$$\dot{f}_\mu = \frac{\partial f_\mu}{\partial \tilde{y}}\dot{\tilde{y}} = \frac{\partial f_\mu}{\partial y}\dot{y} + \frac{\partial f_\mu}{\partial y^0}\dot{y}^0 = 0, \tag{2.13}$$

$$\ddot{f}_\mu = \frac{\partial^2 f_\mu}{\partial y \partial y}\dot{y}\dot{y} + \frac{\partial f_\mu}{\partial y}\ddot{y} =$$

$$\frac{\partial^2 f_\mu}{\partial y \partial y}\dot{y}\dot{y} + 2\frac{\partial^2 f}{\partial y^0 \partial y}\dot{y}\dot{y}^0 + \frac{\partial^2 f}{\partial y^0 \partial y^0}\dot{y}^0\dot{y}^0 + \frac{\partial f_\mu}{\partial y}\ddot{y} + \frac{\partial f_\mu}{\partial y^0}\ddot{y}^0 = 0 \qquad (2.14)$$

The last acceleration relation can be written in a shorter form

$$\frac{\partial f_\mu}{\partial y}\ddot{y} + \frac{\partial f_\mu}{\partial y^0}\ddot{y}^0 = \Phi(y^0, y; \dot{y}^0, \dot{y}), \qquad (2.15)$$

where the composition of function Φ is obvious.

If $\ddot{y}$ from Lagrange's equations of first kind

$$m\ddot{y} = Y + \sum \lambda_\mu \frac{\partial f_\mu}{\partial y}, \qquad (2.16)$$

is included in equation (2.15), it is obtained that:

$$\frac{\partial f_\mu}{\partial y}\sum_{\sigma=1}^{k}\lambda_\sigma \frac{\partial f_\sigma}{\partial y} = m\left(\Phi - \frac{\partial f_\mu}{\partial y^0}\ddot{y}^0\right) - Y\frac{\partial f_\mu}{\partial y}.$$

The solution with respect to unknown multipliers λ_σ shows that the reaction forces of variable constraints do not only depend upon $\overline{y}$ coordinates and $\dot{\overline{y}}$ velocities, but also on $\ddot{y}^0$, as well as on inertia force $-m\ddot{y}^0$ which emerges due to the constraints' change in time.

The constraints in equations (2.10) and (2.12) can be written in the parametric form:

$$\mathbf{r}_\nu = \mathbf{r}_\nu\left(q^0, q^1, \ldots, q^n\right), \quad n = 3N - k \qquad (2.17)$$

where $q = \left(q^1, \ldots, q^n\right)$ are *independent generalized coordinates,* while q^0 is a *rheonomic coordinate* satisfying equation (2.11), that is,

$$q^0 - \tau(t) = 0. \qquad (2.18)$$

By reducing the finite constraints to the parametric form (2.17) the number of differential equations for the constraints' number is also reduced; at the same time, constraints' forces $\boldsymbol{R}$ are eliminated which makes it considerably easier to solve the problem.

The velocities of ν-th material points, according to definition (2.1), can be written in the following form:

$$\mathbf{v}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q^0}\dot{q}^0 + \frac{\partial \mathbf{r}_\nu}{\partial q^1}\dot{q}^1 + \cdots + \frac{\partial \mathbf{r}_\nu}{\partial q^n}\dot{q}^n = \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha}\dot{q}^\alpha \qquad (2.19)$$

where $\dfrac{\partial \mathbf{r}_\nu}{\partial q^\alpha}(q)$ are coordinate vectors that will be marked by two-indices notation $\mathbf{g}_{\nu\alpha}$; index ν denotes the number of the material point, while index α denotes the number of independent coordinates q^α $(\alpha = 0, 1, \ldots, n)$.

For addition with respect to index ν, we use addition sign $\sum\limits_{\nu}$, while for addition with respect to the indices, coordinate α denotes iteration of the same letter in the same expression, as well as both the lower and the upper indices. Vector (2.19), as can be seen, has $n+1$ independent elementary vectors. Accordingly, impulse vector of the ν-th material point of mass m_ν of the observed system can also be represented by the formula

$$p_\nu = m_\nu \mathbf{v}_\nu = m \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \dot{q}^\alpha. \tag{2.20}$$

Scalar multiplication by coordinate vectors $\dfrac{\partial \mathbf{r}_\nu}{\partial q^\beta}$ gives 'coordinate impulse of the ν-th material point:

$$p_{\nu\beta} = m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta} \dot{q}^\alpha, \quad \alpha, \beta = 0, 1, \ldots, n.$$

This is in accordance with the formula for impulse's coordinates (1.25) of one material point. Regarding the fact that $p_{\nu\beta}$ impulses are scalars, it is possible to sum them up:

$$p_\beta := \sum_{\nu=1}^{N} p_{\nu\beta} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta} \dot{q}^\alpha = a_{\alpha\beta} \dot{q}^\alpha, \tag{2.21}$$

from which it can be seen that $a_{\alpha\beta}$ is an inertia tensor of the whole system:

$$a_{\alpha\beta} = \sum_{\nu=1}^{N} m_\nu \frac{\partial \mathbf{r}_\nu}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^\beta} = a_{\alpha\beta}\left(m_1, \ldots, m_N; q^0, q^1, \ldots, q^n\right). \tag{2.22}$$

By means of important relations (2.21) the concept of generalized impulses of the material points' system is introduced. *The generalized impulses appear as linear homogeneous forms of the generalized velocities,* which is in accordance with the basic definition of impulse (2.5). Regarding the fact that the inertia tensor $a_{\alpha\beta}$ determinant is different from zero, it is possible to determine the generalized velocities $\dot{q}^\alpha$ as linear homogeneous combinations of the generalized impulses, namely:

$$\dot{q}^\alpha = a^{\alpha\beta} p_\beta, \tag{2.23}$$

where $a^{\alpha\beta}$ is *contravariant inertia tensor.*

If the constraints do not explicitly depend upon the known functions of time τ, there is no rheonomic coordinate q^0, so that in all the expressions, from (2.16) to (2.22), coordinates $q^0, \dot{q}^0$ and p_0 vanish. The impulse form (2.21) does not change, expect for the fact that indices $\alpha = 0, 1, \ldots, n$ do not assume values from 0 to n, but from 1 to n. In order to facilitate this distinction further on, let Greek indices

$\alpha, \beta, \gamma, \delta$ assume values from 0 to n,$(\alpha, \beta, \gamma, \delta = 0, 1, \dots, n)$, while the Latin ones take i, j, k, l from 1 to n $(i, j, k, l = 1, 2, \dots, n)$. Then it can be written; [1],[3]:

$$\mathbf{v}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q^0}\dot{q}^0 + \frac{\partial \mathbf{r}_\nu}{\partial q^i}\dot{q}^i \tag{2.24}$$

or

$$\begin{aligned} p_i &= a_{0i}\dot{q}^0 + a_{ij}\dot{q}^j \\ p_0 &= a_{00}\dot{q}^0 + a_{0j}\dot{q}^j \\ \dot{q}^i &= a^{i0}p_0 + a^{ij}p_j \\ \dot{q}^0 &= a^{00}p_o + a^{0j}p_j. \end{aligned} \tag{2.25}$$

Therefore, kinetic energy E_k of a rheonomic holonomic system represents a homogeneous quadratic form of generalized velocities, which can be developed into the invariant form

$$2E_k = a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta = a^{\alpha\beta}p_\alpha p_\beta, \quad \alpha, \beta = 1, \dots, n+1, \tag{2.26}$$

what is distinguished from standard noninvariant form

$$2E_k = a_{ij}\dot{q}^i\dot{q}^j + 2b_i\dot{q}^i + c, \quad i, j = 1, \dots, n.$$

In the case of finite geometric constraints, rheonomic coordinate $q^0 = 0$ and formula of kinetic energy obtain known homogeneous quadratic form

$$2E_k = a_{ij}\dot{q}^i\dot{q}^j = a^{ij}p_ip_j, \quad i, j = 1, \dots, n. \tag{2.27}$$

Generalized coordinates $q^1, \dots, q^n$ and generalized impulses $p_1, \dots p_n$ are called somewhere "Hamiltonian coordinates". It is not only formal problem, which will be shown in the following text.

3. NEWTON'S AND HAMILTON'S TASK OF MECHANICS

In the explanation of Definition of centripetal force ([2], p. 27), Newton is stating that " *the task of mathematician is to find such a force, which will keep an observed object at the given orbit, with given velocity, and the other way around: to find such curvilinear way in relation to which the given body is moved from the starting position at the given velocity.*" In the modern literature, this Newton's standpoint is known as I and II, or "direct or inverse task of dynamics."

In general, we are discussing the mixed system of definite and differential equations:

$$m_\nu \frac{d\mathbf{v}_\nu}{dt} = m_\nu \ddot{\mathbf{r}}_\nu = F_\nu, \quad \nu = 1, \dots, N; \tag{3.1}$$

$$f_\mu(\mathbf{r}_\nu, \mathbf{F}_\nu) = 0, \quad \mu = 1, \dots, k \le N, \tag{3.2}$$

which we can use also to determine force..

Example. Two material points are moving, accordingly to the second and third Newton's axiom. The differential equations of motion are:

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_1, \quad m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_2, \quad \mathbf{F}_1 = -\mathbf{F}_2; \tag{3.3}$$

and equation of distance is

$$\boldsymbol{\rho}(t) = \mathbf{r}_2 - \mathbf{r}_1. \tag{3.4}$$

Our task is determine forces. The complete system of 4 equation (3.3) and (3.4). By differentiation of equation (3.4) twice in time

$$\ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1 = \ddot{\boldsymbol{\rho}}, \tag{3.5}$$

is obtained, or accordingly to the equations (3.3),

$$\ddot{\rho} = \frac{\mathbf{F}_2}{m_2} - \frac{\mathbf{F}_1}{m_1} = \mathbf{F}_2\frac{m_1 + m_2}{m_1 m_2}.$$

From here, it follows, [18]:

$$\mathbf{F}_2 = -\mathbf{F}_1 = \frac{m_1 m_2}{m_1 + m_2}\ddot{\boldsymbol{\rho}}, \tag{3.6}$$

as well as

$$F_\rho = \frac{\dot{\rho}^2 - \rho\ddot{\rho} - v_{or}^2}{m_1 + m_2}\frac{m_1 m_2}{\rho}. \tag{3.7}$$

This result, which is first obtained through Lagrange's equations of the first kind, [1], raised the concern at certain numbers of physicians. At the scientific seminars, several participants were of the opinion that the force cannot be determined based on the known motion.In that discussion, it is raised an inspired question: *can the Lagrange's method of indefinite multipliers be used in Hamiltonian differential equations*

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha}, \tag{3.A}$$

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}, \tag{3.B}$$

in which no forces are present, and do not explicit have neither forces nor Lagrange's multipliers of constraints.

The answer to the question was presented at the Congress of Theoretical and applied Mechanics of the Serbian Association for Mechanics 2007, in the paper "Hamilton's inverse problem [5]. The only reaction was by a reader, who was of the opinion that the title was not correct. As the comment has shown that matter was nor understood, we will mention here, in order to present it more clearly, some basic standpoints of the analytical mechanics.

Hamilton's inverse problem. In the papers ([6], pp. 236-237) Hamilton is shows that Lagrange introduced the function of force U, consisted of mass and inter related distances of few points of the system. He further writes:

$$m_i x_i'' = \frac{\partial U}{\partial x_i}, \quad m_i y_i'' = \frac{\partial U}{\partial y_i}, \quad m_i z_i'' = \frac{\partial U}{\partial z_i}. \tag{3.8}$$

After some short transformations and inclusion of function

$$H = E_k - U = E_k(\bar\omega_1, \bar\omega_2, \ldots, \bar\omega_{3N}; \eta_1, \eta_2, \ldots, \eta_{3N}) - U(\eta_1, \eta_2, \ldots, \eta_{3N}) = E_k + E_p,$$

Hamilton is obtains, as he said, a new way of presenting of differential equations of motion of the system of N points that are attracting or repulsing each other:

$$\begin{gathered}
\frac{d\eta_1}{dt} = \frac{\delta H}{\delta \bar\omega_1}; \quad \frac{d\bar\omega_1}{dt} = -\frac{\delta H}{\delta \eta_1}, \\
\frac{d\eta_2}{dt} = \frac{\delta H}{\delta \bar\omega_2}; \quad \frac{d\bar\omega_2}{dt} = -\frac{\delta H}{\delta \eta_2}, \\
\ldots \\
\frac{d\eta_{3n}}{dt} = \frac{\delta H}{\delta \bar\omega_{3n}}; \quad \frac{d\bar\omega_{3n}}{dt} = -\frac{\delta H}{\delta \eta_{3n}},
\end{gathered}$$

where are $\bar\omega$, η, δ Hamilton's symbols for: $\bar\omega = p$, $\eta = y$, $\delta = \partial$.

From that stand point, Hamilton is again emphasis that *a task of mathematical dynamics for a system of N points is, integrate system of $6N$ ordinary differential equations of the first order*, ([6], p. 237). This considerably differs from the two quoted Newton's tasks of Mechanics.

As a difference from Hamilton's understanding of a main task of dynamics, Newton is seeing it in a more general way. In the introduction of the first edition $([2], p.2)$ he emphasized that the rational mechanics is a science on motions produced by forces, and on forces is presented and proved.

In the quoted part, Hamilton is not even using the force as a term in Newton's way. The discussion over it, thus we named a **problem**, e.g. "Hamilton's inverse problem", as there is no other way to define a force using the equations (3.A) and (3.B) in which no forces are present.

Let's also mention that we start from the Newton theory, that is corresponding with Hamilton's method [1]. For the assumption that the function of force $U = -U_p$ is given instead od the force $\mathbf{F}$ and for the assumption tat $m = const.$, the equations (3.8) can be written as:

$$\frac{dp_i}{dt} = -\frac{\partial E_p}{\partial y^i}, \quad i = 1, 2, 3. \tag{3.9}$$

Hamilton's differential equations (3.A) are equivalent only to such systems of equation of motion (3.9), and not to the Newton's general equation (3.8), neither to the equations (3.1). Hamilton's equations (3.A) and (3.B) do not have the same dimension, when equation (1.B) are with dimension of velocity LT^{-1}, and equations (3.A) with dimension of force MLT^{-2}.

Equations (3.8) are the physical base of Hamilton's differential equations (3.A) and (3.B). Transforming the form of equations (3.9) to the forms (3.A) and (3.B) is mathematical formality.

In order to make it more clear let's show it with a simple **example** for motion of one material point. Vector of position of some material point $\mathbf{r} = y^i \mathbf{e}_i$ we will write using the Descartes' coordinates $y \in E^3; \mathbf{e} := (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \in \mathbf{E}_3$, and vector velocity as $\mathbf{v} = \dot{y}^i \mathbf{e}_i$.

With refer to that system of coordinates it will be

$$\mathbf{p} = m\dot{y}^i \mathbf{e}_i \to p_j = \delta_{ij} \dot{y}^i,$$

where δ_{ij} Kronecker's symbols. From here, it follows:

$$\dot{y}^k = \frac{\delta^{ik}}{m} p_i = \frac{p_k}{m}.$$

As the kinetic energy is, according to the definition

$$E_k = \frac{mv^2}{2} = \frac{m\delta_{ij}}{2} \dot{y}^i \dot{y}^j = \frac{\delta^{ij}}{2m} p_i p_j = \frac{1}{2} a^{ij} p_i p_j; \tag{3.10}$$

it is obtained

$$\dot{y}^i = \frac{\partial E_k}{\partial p_i} = \frac{\partial E_k + E_p}{\partial p_i} = \frac{\partial H}{\partial p_i} = \frac{p_i}{m}, \tag{3.11}$$

$$\dot{p}_i = \frac{\partial E_k}{\partial y^i} = \frac{\partial E_k + E_p}{\partial y^i} = -\frac{\partial H}{\partial y^i} = -\frac{\partial E_p}{\partial y^i}. \tag{3.12}$$

The function H is Hamilton's mark ([6], p. 237) for the sum of kinetic and potential energy, e.g.

$$H(y^1, y^2, y^3; p_1, p_2, p_3) = E_k + E_p = \frac{1}{2} a^{ij} p_i p_j + E_p(y). \tag{3.13}$$

Let's observe that relations (3.11) have a vector's structure, as $\dot{y}^i \in R^3(LT)$ are coordinates of tangential vector $\mathbf{v} \in \mathbf{R}_3(LT)$. However. coordinates of impulse $p_i \in R_3(MLT)$ have covariant structure. Thus, differences between equations (3.11) and (3.12) are big. They are even bigger especially at curvilinear coordinate systems $x = (x^1, x^2, x^3)$, and especially generalized coordinates $q^\alpha \in M^n$. However, the function H is a scalar invariant and it should have the same physical dimension of energy in all coordinate systems,

$$dim\, H = ML^2T^{-2}. \tag{3.14}$$

The mentioned fact are not predictable in mechanics, although equations (3.11) have the same form as equations (1.B), and equations (3.12) as general equation (1.A).

It should be previously determined values of indexes $\alpha, \beta, \dots,$ in Hamilton's equations (1.A) and (1.B). For the new condition $f(y^1, y^2, y^3) = const.$ or the new limitation of motion, $f(y^1, y^2, y^3) = 0$ the additional force $\mathbf{R} = \lambda\, grad\, f$ is needed, in equations (3.12) likewise. Those forces can be lost, if the calculation is done according to the homogenous system of equations (1.A) and (1.B).

Let's show it at the simple **example** of motion of the heavy point with mass m, in some plane, e.g. at the condition:

$$f(y^1, y^2, y^3) = ay_1 + by_2 + y_3 = 0; \quad (y^i \equiv y_i). \tag{3.15}$$

The task:

1. To determine magnitude of the force

$$\mathbf{R} = \lambda\, grad\, f, \tag{3.16}$$

that keeps a body in plane (3.15) with Hamilton's homogenous equations and Lagrange's multipliers λ.

2. To solve the inverse problem of Hamilton's homogenous equations and coordinates $(p, q) \in T^*M^2$.

1. Starting of equations (5), i.e. (10) at the condition (19) and Lagrange's equations the first kind, the Hamilton's system of equations is

$$\dot{p}_i = -\frac{\partial H}{\partial y^i} + \lambda \frac{\partial f}{\partial y^i}, \tag{3.17}$$

and

$$\dot{y}^i = \frac{\partial H}{\partial p_i}, \qquad (i, j = 1, 2, 3). \tag{3.18}$$

In this example it is reduced to:

$$\dot{p}_1 = -\frac{\partial H}{\partial y^1} + \lambda \frac{\partial f}{\partial y^1} = -a\lambda,$$

$$\dot{p}_2 = -\frac{\partial H}{\partial y^2} + \lambda \frac{\partial f}{\partial y^2} = -b\lambda,$$

$$\dot{p}_3 = -\frac{\partial H}{\partial y^3} + \lambda \frac{\partial f}{\partial y^3} = -mg + \lambda;$$

$$\dot{y}^i = \frac{p_i}{m};$$

$$f = ay_1 + by_2 + y_3 = 0,$$

because

$$H = E_k + E_p = \frac{\delta^{ij}}{2m} p_i p_j + mgy^3.$$

It follows

$$\ddot{f} = \frac{1}{m}(\dot{p}_1 + \dot{p}_2 + \dot{p}_3) = 0.$$

By moving $\dot{p}_i$ from equations (19) in this relation, it is obtained

$$\lambda = \frac{mg}{1 + a^2 + b^2}.$$

Thus the coordinates of the searched force $\mathbf{R}$ are

$$R_1 = \frac{mga}{1 + a^2 + b^2}, \quad R_2 = \frac{mgb}{1 + a^2 + b^2}, \quad R_3 = \frac{mg}{1 + a^2 + b^2},$$

and the magnitude R of the force $\mathbf{R}$ is $R = mg(1 + a^2 + b^2)^{1/2}$.

2. With homogenous system of equations (1.A) and (1.B), the steps for solving this task are as follows:

$$f(y^1, y^2, y^3) = 0 \to y^3 = -ax - by = -aq^1 - bq^2; \quad (q^1, q^2) \in M^2.$$

$$E_k = \frac{m}{2}(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2) = \frac{m}{2}[(1 + a^2)\dot{(q^1)}^2 + 2ab(\dot{q}^1\dot{q}^2 + (1 + b^2)\dot{(q^2)}^2];$$

$$p_1 = m\big((1 + a^2)\dot{q}^1 + ab\dot{q}^2\big), \quad p_2 = m\Big((ab\dot{q}^1 + (1 + b^2)\dot{q}^2\big);$$

$$\Delta = 1 + a^2 + b^2,$$

$$\dot{q}^1 = \frac{(1 + b^2)p_1 - abp_2}{m\Delta}, \quad \dot{q}^2 = \frac{(1 + a^2)p_2 - abp_1}{m\Delta}.$$

$$H = E_k + E_p = \frac{1}{2m}[(1 + a^2)p_1^2 + 2abp_1p_2 + (1 + b^2)p_2^2] + mg(aq^1 + bq^2).$$

Equations (1.A),

$$\dot{p}_1 = -\frac{\partial H}{\partial q^1} = -\frac{\partial E_p}{\partial q^1} = -mag = Q_1,$$

$$\dot{p}_2 = -\frac{\partial H}{\partial q^2} = \frac{\partial E_p}{\partial q^2} = -mbg = Q_2,$$

are clearly showing that generalized forces Q_α are not considering force R.

As an **example**, Hamilton is observing a system of two points, with a known function of force, ([6], pp. 199-200),

$$U = m_1 m_2 f(r)$$

where:
$$r = \big((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\big)^{1/2}$$
For ([6], pp. 207-212)
$$f(r) = \frac{1}{r}$$
Hamilton is observing motion of planets or comets, which are obedient to Newton,s gravitation law. Such problem is related to the integration of differential equations of motion. In this case, when the function of force is given:

$$U = m_1 m_2 f(r) = m_1 m_2 \big((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\big)^{-1/2}$$

a force can be determined without equation (1.A) and (1.B). In:

$$F_{x_1} = \frac{\partial U}{\partial x_1} = m_1 m_2 \frac{x_1 - x_2}{r^3}, \ldots, F_{z_1} = \frac{\partial U}{\partial z_1} = m_1 m_2 \frac{z_1 - z_2}{r^3},$$

the needed force is obtained

$$F_1 = -\big(F_{x_1}^2 + F_{y_1}^2 + F_{z_1}^2\big)^{1/2} = -\frac{m_1 m_2}{r^2}.$$

If the function U is known; also the force $\mathbf{F}$ is known, we see the inverse problem as: *a determination of force based on the known attributes od motion - position, velocity, or position and impulse of motion.*

Our approach, in the concrete **Hamilton's example** of motion of two material points of masses m_1 and m_2, we determine the force F_1, with which the body of the mass m_1 is acting at the body of the mass m_2 at the equation

$$f = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - \rho^2(t) = 0. \tag{3.19}$$

Differential equations of motion (3.17) for this example are:

$$m_1 \ddot{x}_1 = \lambda \frac{\partial f}{\partial x_1}, \quad m_1 \ddot{y}_1 = \lambda \frac{\partial f}{\partial y_1}, \quad m_1 \ddot{z}_1 = \lambda \frac{\partial f}{\partial z_1};$$

$$m_2 \ddot{x}_2 = \lambda \frac{\partial f}{\partial x_2}, \quad m_2 \ddot{y}_2 = \lambda \frac{\partial f}{\partial y_2}, \quad m_2 \ddot{z}_2 = \lambda \frac{\partial f}{\partial z_2}.$$

With equations (3.19), the unknown multiplier of constraint, can be determined:

$$\dot{f} = 2[(x_2 - x_1)(\dot{x}_2 - \dot{x}_1) + (y_2 - y_1)(\dot{y}_2 - \dot{y}_1) + (z_2 - z_1)(\dot{z}_2 - \dot{z}_1) - \rho\dot{\rho}] = 0;$$

$$\ddot{f} = 2[(\dot{x}_2 - \dot{x}_1)(\ddot{x}_2 - \ddot{x}_1) + (\dot{y}_2 - doty_1)(\ddot{y}_2 - \ddot{y}_1) + (\dot{z}_2 - \dot{z}_1)(\ddot{z}_2 - \ddot{z}_1) - \dot{\rho}^2 + \rho\ddot{\rho}] = 0$$

By substituting $\ddot{y}^i$ from the equation of motion in the previous equation it is obtained
$$\lambda = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{m_1 + m_2} \frac{m_1 m_2}{\rho^2}.$$

And as

$$F_1 = -\lambda\sqrt{(\frac{\partial f}{\partial x_1})^2 + (\frac{\partial f}{\partial y_1})^2 + (\frac{\partial f}{\partial z_1})^2},$$

the force is obtained

$$F_1 = -\chi\frac{m_1 m_2}{\rho}, \tag{3.20}$$

where:

$$\chi = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{m_1 + m_2}.$$

If the observed material points are planets of the Sun system and the Kepler's Laws are taken in consideration, i.e.

$$\rho = \frac{p}{1 + e\cos\theta}, \quad \rho^2\dot{\theta} = C = \frac{2ab\pi}{T},$$

the formula (3.20) is obtained

$$F = -\frac{4\pi^2 a^3}{(m_1 + m_2)T^2}\frac{m_1 m_2}{\rho^2} = -k\frac{m_1 m_2}{\rho^2}, \tag{21}$$

where

$$k = \frac{4\pi^2 a^3}{(m_1 + m_2)T^2}$$

is a factor of proportionality, known also as *gravitational constant.*

Let's discuss now the solutions of that task using the Hamilton's variables: $x, y, z; p_x, p_y, p_z$. For the previous example of two bodies motion at the distance $\rho(t)$, differential equations of motion are reduced to::

$$\dot{x}_1 = \frac{p_{1x}}{m_1}, \quad \dot{y}_1 = \frac{p_{1y}}{m_1}, \quad \dot{z}_1 = \frac{p_{1z}}{m_1};$$

$$\dot{x}_2 = \frac{p_{2x}}{m_2}, \quad \dot{y}_2 = \frac{p_{2y}}{m_2}, \quad \dot{z}_2 = \frac{p_{2z}}{m_2}.$$

$$\dot{p}_{1x} = \lambda\frac{\partial f}{\partial x_1}, \quad \dot{p}_{1y} = \lambda\frac{\partial f}{\partial y_1}, \quad \dot{p}_{1z} = \lambda\frac{\partial f}{\partial z_1}; \tag{3.23}$$

$$\dot{p}_{2x} = \lambda\frac{\partial f}{\partial x_2}, \quad \dot{p}_{2y} = \lambda\frac{\partial f}{\partial y_2}, \quad \dot{p}_{2z} = \lambda\frac{\partial f}{\partial z_2}. \tag{3.24}$$

The conditions for velocities and accelerations are

$$\dot{f} = 2[(x_2 - x_1)(\frac{p_{2x}}{m_2} - \frac{p_{1x}}{m_1}) + (y_2 - y_1)(\frac{p_{2y}}{m_2} - \frac{p_{1y}}{m_1}) + (z_2 - z_1)(\frac{p_{2z}}{m_2} - \frac{p_{1z}}{m_1}) - \rho\dot{\rho}] = 0;$$

$$\ddot{f} = 2[(\frac{p_{2x}}{m_2} - \frac{p_{1x}}{m_1})^2 + (\frac{p_{2y}}{m_2} - \frac{p_{1y}}{m_1})^2 + (\frac{p_{2z}}{m_2} - \frac{p_{1z}}{m_1})^2 + (x_2 - x_1)(\frac{\dot{p}_{2x}}{m_2} - \frac{\dot{p}_{1x}}{m_1}) +$$

$$(y_2 - y_1)\big(\frac{\dot p_{2y}}{m_2} - \frac{\dot p_{1y}}{m_1}\big) + (z_2 - z_1)\big(\frac{\dot p_{2z}}{m_2} - \frac{\dot p_{1z}}{m_1}\big)] = 2(\dot\rho^2 + \rho\ddot\rho).$$

By substituting of the first derivative of impulse $\dot p_i$ from the equation of motion (3.23) and (3.24) into the previous relation, it is obtained:

$$v_{or}^2 + \lambda[(x_2 - x_1)\big(\lambda\frac{\partial f}{\partial x_2}\frac{1}{m_2} - \frac{\partial f}{\partial x_1}\frac{1}{m_1}\big) + \ldots + (z_2 - z_1)\big(\frac{\partial f}{\partial z_2}\frac{1}{m_2} - \frac{\partial f}{\partial z_1}\frac{1}{m_1}\big)] - \dot\rho^2 - \rho\ddot\rho = 0,$$

or shorter

$$v_{or}^2 + \lambda\frac{m_1 + m_2}{m_1 m_2}\rho^2 - \dot\rho^2 - \rho\ddot\rho = 0, \tag{3.25}$$

as

$$\big(\frac{p_{2x}}{m_2} - \frac{p_{1x}}{m_1}\big)^2 + \big(\frac{p_{2y}}{m_2} - \frac{p_{1y}}{m_1}\big)^2 + \big(\frac{p_{2z}}{m_2} - \frac{p_{1z}}{m_1}\big)^2 = v_{or}^2.$$

Thus, the formula's (3.20) are reobtained, through the more complicated way.

The mentioned examples of motion of two bodies clearly show that Hamilton is discussing this task, starting from the known potential energy or function of force $U = 1/\rho$, while at Newton, the force is present in the start-up equations (3.1).

The obtained formula (3.20) is more general from the formula (3.21). Its importance is shown in the best way at the example of determining of the force of the Sun's and the Earth's acting on the Moon.

4. DYNAMICAL PARADOX IN THEORY OF LUNAR MOTION

Standard approach to the problem. In the broad print book (8) on the 64. page, the question raised: Why the Moon does not fall on the Sun? "The question may look naive", the author says, "however when the readers found out that the Sun is attracting the Moon with the higher force than the Earth is, they shown the superior. "Using the simple calculation, the author is showing that the Sun's attraction is twice bigger that the Earth's. In the book of a higher mathematical rank ([9], p. 149), it is more precisely calculated that the Sun's force is 2.5 time bigger then the Earth's one. Such dynamic paradox is obtained if the mentioned forces are calculated using the well known formulas of the size of the "Newton's universal force of gravitation" (3.21),

$$F_\odot = -k\frac{M_\odot m}{\rho_\odot^2}, \tag{4.1a}$$

where $M_\odot$ mass of Sun, $F_\odot$ Sun's force which acts to the Moon of mass m. The force of Earth, which mass $M_\oplus$, which acting to the Moon is

$$F_\oplus = -k\frac{M_\oplus m}{\rho_\oplus^2}, \tag{4.1b}$$

The relation of their magnitudes is

$$\frac{F_\odot}{F_\oplus} = \frac{M_\odot}{M_\oplus}\cdot\frac{\rho_\oplus^2}{\rho_\odot^2}.$$

For known numerical values [10],[11],[12] of masses $M_\odot, M_\oplus$ and distance $\rho_\odot$ and $\rho_\oplus$ we can say that

$$F_\odot > 2F_\oplus.$$

In the book ([1], p.149) we find:

$$F_\odot = 2.5\,F_\oplus.$$

But that result contradicts to the aspects in the nature and also to the laws of classical mechanics. In the book "Physics and Astronomy of the Moon", ([2], p. 9) is written: "The lunar theory-one of the biggest problems of the celestial mechanics-and developed differently than other planet theories." At the seminar for mechanics of Mathematical Faculty of Belgrade University, that was held on 4 March 2003 astrophysicist A.Tomić noticed that it is possible, by the formula (3.20) from the book Vujičić [1], and paper [19], to solve the problem of the centuries, problem of the paradox in theory of the Lunar orbit.

The author of this paper suggests the solution of Dynamic of paradox in theory of Moon's motion, from the point of view of the classical mechanics of two material point. Analytical proofs are closed to the facts that can be found in the scientific literature. Digression from completely true facts, if those facts exist at all, does not influence the author's conclusion - that the force of Earth's attraction of the Moon is larger than the force of the Sun. We have to start from the formula (3.20) considering the fact that the eccentricity of the Moon's and the Earth's path is small, so we have to considered the motion along the circular path in the ecliptic plane. We suggest one solution for dynamic paradox of theory of the Moon's motion from the point of view of the classical mechanics. Discrepancy from exact fact, if theory really exist, does not is influence the author's conclusion - that the Earth's force of the attraction is larger than the force of the Sun.

Tending to prove this, the author thinks that it is necessary to mention some deportment of the analysis and classical kinematics.

The second derivative $\ddot{\mathbf{r}}$ of radius vector $\mathbf{r}(t)$ with time t, in the natural system of the coordinates (see for example [15], p. 34) is

$$\frac{d^2\mathbf{r}}{dt^2} = K(\frac{ds}{dt})^2\mathbf{n}_0 + \frac{d^2s}{dt^2}\boldsymbol{\tau}_0, \tag{4.2}$$

where: K is the curvature of the curve in the point t; v is magnitude of the velocity vector, and $dv/dt = d^2s/dt^2$ is tangential acceleration in that point. We can, also, find this in kinematics, where t is time, as independent variable, ([16], p.30),

$$\ddot{\mathbf{r}} = \frac{dv}{dt}\boldsymbol{\tau}_0 + \frac{v^2}{R_k}\mathbf{n}_0; \quad R_k = \frac{1}{K}, \tag{4.3}$$

where v^2/R_k is magnitude of normal acceleration.

So we can say that the magnitude od acceleration in any point of the path is

$$|\ddot{\mathbf{r}}| = \pm\sqrt{(\frac{dv}{dt})^2 + (\frac{v^2}{R_k})^2}. \qquad (4.4)$$

By multiplication the relation (4.2) or (4.3) with the unit vector $\mathbf{r}_0 = \mathbf{r}/r$, we get radial acceleration

$$w_\rho = \frac{\dot{r}^2 + r\ddot{r} - v^2}{r} = \frac{dv}{dt}\cos\varphi + \frac{v^2}{R_k}\sin\varphi; \quad R_k = \frac{1}{R_k}, \qquad (4.5)$$

where the angle φ is made by the tangent and the radius vector. The relations (4.3) and (4.4) show us that acceleration vector is in the plain, as those basic vectors ρ_o and $\mathbf{n}_o$. In relation to plane coordinate system $\rho, \theta; \rho_o, \theta_o$ the velocity has radial $\dot{\rho}$ and transversal component $\rho\dot{\theta}$

Acceleration radial has the form

$$w_\rho = \frac{D\dot{\rho}}{dt} = \ddot{\rho} - \rho\dot{\theta}^2 \qquad (4.6)$$

because it is

$$v^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2. \qquad (4.7)$$

Vice versa is valid proof, also. It is well known that acceleration radial responds to the covariant derivation by the time from the radial velocity $\dot{\rho}$, e.g.

$$w_\rho = \frac{D\dot{\rho}}{dt} = \ddot{\rho} - \rho\dot{\theta}^2 = \ddot{\rho} - \frac{\rho^2\dot{\theta}^2}{\rho}.$$

Considering (4.7), where $\rho^2\dot{\theta}^2 = v^2 - \dot{\rho}^2$ is from, we get the

$$w_\rho = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{\rho}. \qquad (4.8)$$

In the literature of mechanics and physics ([9], p. 194), is shown what is the size of the radial acceleration at the circle movement on the different height above the Earth by the formula

$$\gamma = g\frac{R^2}{\rho^2},$$

and by the formula

$$\gamma^* = \frac{v^2}{\rho},$$

and that can be used, on satellites on the circle paths around the Earth.

Altetude	Velocity	Acceleration	Acceleration
H km	v km/s	γ	γ^*
0	7,91	981,0	982,3
100	7,84	948,9	950,0
1000	7,35	732,1	733,0
10000	4,93	148,4	148,4
100000	1,94	3,5	3,5
384400	1,02	0,002693	0.002670

The last type of this table is in the relation from the moon's motion on the distance $\rho = 384400\,\text{km}$ to the center of the Earth.

Two bodies. It is proved that the gravity force of two bodies, which have volumes m_1 and m_2, and which move according to the Newton's axioms, on the distance $\mathbf{r}_2 - \mathbf{r}_1 = \boldsymbol{\rho}$ between their's center of masses can be written, in the form (3.6). Projection of that formula (3.6) on the axis of the vector $\boldsymbol{\rho}_o$ can be obtained as scalar product the equation (3.6) and vector

$$\boldsymbol{\rho}_o = \frac{\boldsymbol{\rho}}{\rho}$$

in the form (3.7), where $v_o r$ is the velocity of one body for this problem,

$$v_{or} = v_2 - v_1. \tag{4.9}$$

In relation to Descartes's system of coordinates that is shown as

$$v_{or} = \sqrt{(\dot{x}_2 - \dot{x}_1)^2 + \cdots + (\dot{z}_2 - \dot{z}_1)^2} \tag{4.10}$$

We have to mention that $F = 0$, if it is $\frac{D\dot{\rho}}{dt} = 0, \longrightarrow \ddot{\rho} = \rho\dot{\theta}^2$ and also it is

$$\frac{D\dot{\rho}}{dt} = -\rho\dot{\theta}^2, \longrightarrow F = -\frac{m_1 m_2}{m_1 + m_2}\frac{v^2}{\rho}, \tag{4.11}$$

if $\rho = const.$

A possible solution of the problem. Average velocity of the Earth, [10],[11], $v_\oplus = 29.84\,\text{km/s}$; distance between the Earth and the Moon is $\rho = 384400\,\text{km}$. The velocity of the Moon is $v = 1.02\,\text{km/s}$; mass of the Earth is $M_\oplus = 5.97 \times 10^{24}\,\text{kg}$ and mass of the Moon $m = 0.0739 \times 10^{24}\,\text{kg}$. By changing in the formula we find out that gravity force by means of which the Earth attracts the Moon with is the same as

$$\mathcal{F}_\oplus = -\frac{M_\oplus m}{M_\oplus + m}\frac{v_{or}^2}{\rho_\oplus} = 0.987839 \cdot \frac{(v_\oplus + 1.02 - v_\oplus)^2}{384400} = \mathbf{0.0026736}\, m, \tag{4.12}$$

because

$$\frac{M_\oplus}{M_\oplus + m} = 0.987839878.$$

We can obtain proper force of the Sun in the same way. The formula of the force (4.11) between the Sun and the Moon is

$$\mathcal{F}_\odot = -\frac{M_\odot m}{M_\odot + m}\frac{v_{or}^2}{\rho_\odot}, \tag{4.13}$$

where is

$$\frac{M_\odot}{M_\odot + m} = 0.999999.$$

In the book ([8], p.167) is written: "All stars in our group of stars, including the Sun, move toward each other at mean deviation velocity of 30 km/s, e.g. that is the velocity of our planet along it's path."

In the book ([21], p. 80 and 383) of the advanced mathematical level, the velocity $v_\odot$ of the Sun in space is more precisely defined. In the system of galactic coordinates of *apex* $l = 24^o, b = 22^o$, the generally adopted velocity of Sun is $v_\odot = 20$ km/s.

In the scientific literature: Kulikovskij, ([22], p.78), [23] we can see that those speeds respond to the velocity of the Sun $\mathbf{v}_\odot = \mathbf{29.6}$ km/s to the centroid of 1214 stars to the galactic apex $L = 59^o, B = 26^o$. For standard motion the velocity of the Sun is **19.5** km/s; $L = 56^o, B = 23^o$.

Let's calculate the Sun's force gravity $\mathcal{F}_\odot$ for the last two velocities:

$$v_{or,1} = v_M - v_S = (v_\oplus + 1.02) - 29.6 = 29.8 + 1.02 - 29.6 = 1.22\, km/s;$$

$$v_{or,2} = v_M - v_S = (v_\oplus + 1.02) - 19.5 = 29.8 + 1.02 - 19.5 = 11.32\, km/s.$$

On the basic formula (4.13) it is obtained:

$$\mathcal{F}_{\odot 1} = 0.999999\frac{1.22^2}{149.6 \times 10^6} m = \mathbf{9.9491968} \times \mathbf{10^{-6}}\, m,$$

$$\mathcal{F}_{\odot 2} = 0.999999\frac{11.32^2}{149.6 \times 10^6} m = \mathbf{0.85656688} \times \mathbf{10^{-3}}\, m.$$

It is obvious that the gravity force $\mathcal{F}_\oplus$ of the Sun is larger than the force of the Sun;

$$\mathcal{F}_\oplus = 268.670\mathcal{F}_{\odot 1},$$

and

$$\mathcal{F}_\oplus = 3.122\mathcal{F}_{\odot 2},$$

or generally

$$\mathcal{F}_\oplus > \mathcal{F}_\odot.$$

Elliptic motion. Considering the fact that elliptic path of the Moon is similar to the circular path, this approach is enough for rejection "the paradox of the Moon's motion". Nevertheless, we are going to show this for elliptic path, too. For keeping the Moon on the elliptic path we need to have radial acceleration, along the radius ρ between the mass centers of the earth and the Moon is zero, or

$$w_\rho = \ddot{\rho} - \rho\dot{\theta}^2 = 0,$$

e.g. that is

$$\ddot{\rho} = \rho\dot{\theta}^2, \tag{4.14}$$

because transversal acceleration, considering the third Newton's axiom is equal to zero,

$$\rho\ddot{\theta} + 2\rho\dot{\theta} = \frac{1}{\rho}\frac{d}{dt}\rho^2\dot{\theta} = 0.$$

It follows

$$\rho^2\theta = C = \frac{2\pi ab}{T},$$

where T sidereal period of Moon's revolution. In continuation from equation (4.14) is obtain:

$$\gamma := \ddot{\rho} = \rho\frac{4\pi^2a^2b^2}{\rho^4T^2}_{|\rho=a} = \frac{4\pi^2}{T^2}a(1-e^2).$$

As the eccentricity of the Moon $e = 0.0549$, we get for acceleration $\gamma_\oplus = 0.00271366$, that is close to $\gamma_\oplus = 0.002709937$, that we received for the circular motion. Considering the fact that the eccentricity of the Earth is $e = 0.0168$, less than the eccentricity of the moon, and the better correspondence with the acceleration 0.000006959 is achieved. So, the problem of the paradox of the Moon's motion, is solved.

The Newton's task of determination of force, by which the Sun and the Earth are acting to the Moon's motion, using the classical perturbation theory, belongs to the problems of three bodies; such solution is proposed at the other place. Thus it should be taken into consideration that also in the perturbation theory there are disagreements. Let's then present our contribution to that theory.

5. ON STABILITY OF MOTION

In the referential literature about bodies' motion the differential equations of disturbed motion do not always imply the same thing, regardless of the fact that the term is general. In the general theory of planet disturbances, these are, in the most general sense, differential equations of motion (See, for instance, ([28, p. 53])

$$m_\nu\ddot{\mathbf{r}}_\nu = \mathbf{F}_\nu + \mathbf{G}_\nu \tag{5.1}$$

which the disturbance forces are added to. While describing the system's motion by means of equations (3.A) and (3.B), when forces Q_i^* are absent, the equations of disturbed motion are found in the form of variation, [25]:

$$\frac{d}{dt}\delta p_i = -\frac{\partial^2 H}{\partial q^j\partial q^i}\delta q^j - \frac{\partial^2 H}{\partial p_j\partial q^i}\delta p_j,$$

$$\frac{d}{dt}\delta q^i = \frac{\partial^2 H}{\partial q^j \partial p_i}\delta q^j + \frac{\partial^2 H}{\partial p_j \partial p_i}\delta p_j. \tag{5.2}$$

In the motion stability theory, the differential equations of disturbed motion are reduced to the general form:

$$\frac{d\xi}{dt} = f(t,\xi), \quad \xi \in R^n \tag{5.3}$$

Equations (5.1) essentially differ from the other given ones; they serve as the basis for elaborating the whole theory of the planet disturbances. All the other above-given systems of differential equations of disturbances are formed of the basic differential equations of motion by being developed into the degree order or by varying the functions and their derivatives.

In [26] it has been proved that the vector projection variation is not equal to the variation vector projection; thus, instead of equations (5.2) *the covariant differential equations of disturbance* are derived in the form

$$\frac{D\eta_\alpha}{dt} = \psi_\alpha(t,\eta,\xi), \tag{5.4}$$

$$\frac{D\xi^\beta}{dt} = a^{\alpha\beta}\eta_\alpha. \tag{5.6}$$

Invariant Criterion of Motion Stability. The concept of the *invariant criterion* implies general measurement standard in all the coordinate systems for estimating stability of some undisturbed mechanical system's motion. As such, it comprises stability of the equilibrium position and state, stability of stationary motions and, in general, of motion of mechanical systems whose disturbance equations are of coordinate shape [1] and [27].

If for the differential equations of disturbance (6.21)–(6.22) *there is such a positively definitive function* W *of disturbance* $\xi^0,\ldots,\xi^n$ *and time* t *that the expression is*

$$\frac{\partial W}{\partial t} + a^{\alpha\beta}\left(\Psi_\alpha + \frac{\partial W}{\partial \xi^\alpha}\right)\eta_\beta \le 0 \tag{5.6}$$

smaller or equal to zero, the undisturbed state of the mechanical system's motion is stable.

If neither forces $\mathbf{F}^*_\nu$ and $\mathbf{F}$ from relations (5.6) nor differences $\mathbf{F}^*_\nu - \mathbf{F}_\nu$ depend of time t on position $\mathbf{r}$ and velocity $\mathbf{v}$, function Ψ_γ will also be explicitly independent of t. Then function W should also be looked for only in its dependence on disturbances $\xi^0,\xi^1,\ldots,\xi^n$, that is, $W = W(\xi^0,\xi^1,\ldots,\xi^n)$, so that expressions (5.6) and (5.9) are reduced to

$$a^{\alpha\beta}\left(\Psi_\alpha + \frac{\partial W}{\partial \xi^\alpha}\right)\eta_\beta \le 0. \tag{5.10}$$

If the mechanical system's constraints do not depend on time, $q^0, \xi^0, \eta_0, \Psi_0$, vanish, so that expression (5.6), that is (5.9), is reduced to

$$\frac{\partial W}{\partial t} + a^{ij}\left(\Psi_i + \frac{\partial W}{\partial \xi^i}\right)\eta_j \leq 0,$$

while expression (5.10) is reduced to

$$a^{ij}\left(\Psi_i + \frac{\partial W}{\partial \xi^i}\right)\eta_j \leq 0$$

where Ψ_i and W do not depend on ξ^0 and η^0.

All the expressions of the previously given criterion for the equilibrium state stability appear as consequences of expression (5.9) if ξ and η are regarded as disturbances of equilibrium state q and p.

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Sent: Tuesday, June 03, 2008 11:56 AM

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