THE GOVERNING PARAMETER METHOD AS A
GENERAL PROCEDURE FOR STRESS INTEGRATION
OF INELASTIC CONSTITUTIVE RELATIONS
WITHIN INCREMENTAL NONLINEAR ANALYSIS

Miloš Kojić

Abstract. The governing parameter method (GPM), as a general implicit stress integration procedure of the constitutive relations in inelasticity, is presented. A short review of various computational algorithms is given, followed by the formulation of the GPM and application of the GPM to a general isothermal plasticity material model. A basic relations showing the extension of the GPM to large strains is also presented, including two solved numerical examples. In the concluding remarks are summarized the basic features of the GPM and inelastic material models to which the GPM is applicable.

1. Introduction

We briefly present historical review of procedures for stress integration, with the emphasis on the implicit procedures, because they have been favored in modern finite element analysis. One of these implicit procedures is the governing parameter method. A detailed review of the stress integration procedures in inelastic analysis of material deformations is given in Kojić (2002a), Kojić and Bathe (2004).

An iterative scheme, which can be related to the governing parameter method for solving elastic-plastic problems, called the method of successive elastic solutions, was proposed by Ilyushin (1943). Mendelson (1968) formulated this method for von Mises plasticity in a form suitable for numerical methods and used it within finite difference numerical technique.

Another group of procedures relies on the formulation of a tangent material elastic-plastic matrix and represents the explicit schemes. Various forms of stress calculation using this approach have been proposed. We cite two of these approaches, the tangent stiffness-radial return method (Marcal 1965), and the secant stiffness method (Rice and Tracey 1973). These calculation schemes were abundantly used in the past, e.g., Desai and Siriwardane (1984), Chen and Mizuno
(1990). The two most important shortcomings of these schemes are as follows. Firstly, the accuracy of these methods is relatively low, and secondly, the calculated elastic-plastic matrix does not provide a quadratic convergence within the overall equilibrium iterations, as obtained with the algorithmic consistent tangent matrix $t^{+\Delta t}C^{EP}$ obtained when using the implicit schemes.

The development of implicit schemes for stress integration started by Wilkins (1964). The procedure consists of two steps: a) calculation of the elastic solution for the time step (elastic predictor), and b) the radial return to the yield surface (plastic corrector). This approach was further generalized by a number of authors, leading to a class of return mapping procedures. We cite some of the references: Ortiz et al. (1983), Simo and Taylor (1985), Simo and Hughes (1998).

Independent of these developments of the implicit schemes, Kojić and Bathe formulated the “effective-stress-function” algorithm (Kojić and Bathe 1987a,b). The algorithm was applied to various physical conditions (three-dimensional, two-dimensional plane stress, shell, beam and pipe conditions), for von Mises general isotropic and kinematic hardening and common creep laws. The algorithm was extended to the governing parameter method (GPM) in Kojić (1996a), and implemented to various material models (e.g., Kojić 1993, Kojić et al. 1995a, 1995b, 1995c, 1996b, 2002b, Kojić and Bathe 2004). The governing parameter method reduces to solving one nonlinear equation with respect to the governing parameter. A number of authors also calculated the stresses for various inelastic material models through solving one nonlinear equation, e.g., Simo and Taylor (1985), Weber and Anand (1990), Borja and Lee (1990).

A central point in any inelastic finite element analysis is the calculation of the stresses in an effective manner. The calculation must be robust, accurate and efficient (Kojić and Bathe 2004). In the next section we present in some detail the governing parameter method as an effective numerical procedure for the stress calculation in metal plasticity.

2. Formulation of the governing parameter method (GPM)

We assume that at a material point the stress/strain state at time “$t$” is known. An incremental analysis of the body deformation is performed, with time (load) step $\Delta t$, and we suppose that the total strains at the end of the load step are known. The known quantities at a material point are:

$$t\sigma, \ t e, \ t e^{IN}, \ t \beta, \ t^{+\Delta t}e$$

where $t\sigma$, $t e$ and $t e^{IN}$ are stresses, strains and inelastic strains at time $t$; $t \beta$ are internal variables at time $t$ used to describe the history of inelastic deformation, and $t^{+\Delta t}e$ are the total mechanical strains at the end of the load step. The number of internal variables $\beta$ depends on the material model and on the type of inelastic deformation; for example, for a general 3-D elastic-plastic deformation of an initially isotropic metal with mixed hardening, the internal variables are the accumulated effective plastic strain and the components of the position of the yield surface tensor.
Known quantities:
\[ t_\sigma, t_e, t_e^{IN}, t_\beta, t+\Delta t e \]

Unknown quantities:
\[ t+\Delta t_\sigma, t+\Delta t e^{IN}, t+\Delta t_\beta \]

Step 1. Express all unknowns in terms of one unknown parameter \( p \) and known quantities

\[ t+\Delta t_\sigma \left( t_\sigma, t_e, t_e^{IN}, t_\beta, t+\Delta t e, p \right) \]
\[ t+\Delta t e^{IN} \left( t_\sigma, \ldots, p \right) \]
\[ t+\Delta t_\beta \left( t_\sigma, \ldots, p \right) \]

Step 2. Form a function \( f(p) \) and solve the governing equation

\[ f(p) = 0 \]

Step 3. Substitute the solution \( t+\Delta t p \) of the governing equation in (b) to determine the unknowns in (a)

Table 1. Computational steps for stress integration according to the governing parameter method

The task of the stress integration is to determine the stresses \( t+\Delta t_\sigma \), inelastic strains \( t+\Delta t e^{IN} \) and internal variables \( t+\Delta t_\beta \) at time \( t + \Delta t \). Therefore, the unknowns are

\[ t+\Delta t_\sigma, t+\Delta t e^{IN}, t+\Delta t_\beta \]

We use an implicit integration procedure corresponding to the Euler backward method. The basic steps are as follows:

1) Express all unknown variables in terms of one parameter \( p \).
2) Form a function of \( p \) whose zero provides the solution for the governing parameter, that is \( t+\Delta t p \).
3) Calculate the unknown variables using \( t+\Delta t p \).

The computational steps are summarized in Table 1.

In order to provide high convergence rate within equilibrium iterations, it is necessary to determine the tangent constitutive relations \( t+\Delta t C_{ij} \) at the end of
the time step, consistent with the stress integration algorithm. We call $t^{+\Delta t}C$ the consistent tangent constitutive matrix, for which

$$
t^{+\Delta t}C = \frac{\partial t^{+\Delta t}\sigma}{\partial t^{+\Delta t}e}.
$$

Since the stresses $t^{+\Delta t}\sigma_{ij}$ are only functions of strains $t^{+\Delta t}e_{ij}$ and $t^{+\Delta t}p$, (the governing parameter is also a function of the strains), we derive this matrix using the chain rule for calculation of the derivatives in (3) as

$$
t^{+\Delta t}C = \frac{\partial t^{+\Delta t}\sigma}{\partial t^{+\Delta t}e}
\mid_{p=\text{const}} + \frac{\partial t^{+\Delta t}\sigma}{\partial t^{+\Delta t}p} \frac{\partial t^{+\Delta t}p}{\partial t^{+\Delta t}e}
$$

where the first term on the right hand side assumes differentiations under the condition $p = \text{const}$.

The derivatives of the governing parameter $t^{+\Delta t}p$ with respect to the strains are obtained from the condition that the governing equation $f(p) = 0$ must be satisfied at any time of the material deformation. Therefore, we have that $df = 0$ and

$$
\left(\frac{\partial f^T}{\partial \sigma} \frac{\partial \sigma}{\partial p} + \frac{\partial f^T}{\partial e^{\text{TENS}}} \frac{\partial e^{\text{TENS}}}{\partial p} + \frac{\partial f^T}{\partial \beta} \frac{\partial \beta}{\partial p} + \frac{\partial f}{\partial p}\right) \frac{\partial t^{+\Delta t}p}{\partial t^{+\Delta t}e} + \frac{\partial t^{+\Delta t}f}{\partial t^{+\Delta t}e}
\mid_{p=\text{const}} = 0
$$

where matrix notation is used. From this equation we determine the derivatives $\frac{\partial t^{+\Delta t}p}{\partial t^{+\Delta t}e}$.

### 3. Application of the GPM to Time Independent Plasticity Models

We consider the class of time independent plasticity models (Kojić and Bathe 2004) for which the yield criterion can be expressed in terms of the deviatoric stresses $t^{+\Delta t}S$, the mean stress $t^{+\Delta t}\sigma_{m}$, and internal variables $t^{+\Delta t}\beta$ and assume the associated flow rule. According to the Euler backward method we have the following approximation for the increment of the plastic strains $\Delta e^{P}$ in the time step

$$
\Delta e^{P} = \int_{t}^{t^{+\Delta t}} \left(\frac{\partial f_{y}}{\partial \sigma} \lambda\right) dt = \Delta \lambda \frac{\partial t^{+\Delta t}f_{y}}{\partial t^{+\Delta t}\sigma}
$$

where $\Delta \lambda$ is a positive scalar corresponding to the time step $\Delta t$. This is one of the major approximations in the algorithm.

For the class of material models considered here, the yield condition at the end of the time step can be written in the form

$$
t^{+\Delta t}f_{y}(t^{+\Delta t}\sigma, t^{+\Delta t}e) = t^{+\Delta t}f_{y}(t^{+\Delta t}S, t^{+\Delta t}\sigma_{m}, t^{+\Delta t}\beta) = 0
$$

Figure 1 shows schematically two configurations of a generic body $B$, at the start and the end of the time increment $\Delta t$, and the corresponding yield surfaces in the stress space at a material point $M$. Note that here the time step $\Delta t$ represents actually a load step with the strain increment $\Delta e$. The material point $M$ moves
The first step in the return mapping concept is to calculate the trial elastic state. We assume that only elastic deformations occurred in the time step and calculate the stresses $t + \Delta t \sigma^E$ according to the elastic stress-strain relations

$$ t + \Delta t \sigma^E = C^E (t + \Delta t \epsilon'' - \epsilon'') $$

where $C^E$ is the elasticity matrix, and $t + \Delta t \epsilon''$ is the trial elastic strain at the end of the time step (with no plastic flow in time step $\Delta t$). A nonlinear elastic constitutive law may be used to find $t + \Delta t \sigma^E$ corresponding to the strain increment $\Delta \epsilon$. For simplicity of presentation, we use here $C^E$ to be a constant matrix. Then we calculate the yield function $t + \Delta t f_y^E$

$$ t + \Delta t f_y^E = t + \Delta t f_y \left( t + \Delta t \sigma^E, \epsilon' \right) $$

and check for yielding in the time step. If

$$ t + \Delta t f_y^E \leq t f_y $$

the deformation in the time step is elastic and $t + \Delta t \sigma^E$ is the solution. If

$$ t + \Delta t f_y^E > t f_y $$

the plastic deformation of body $B$ during time increment

$\Delta t$. a Physical space; b Stress space

from point $tM$ to point $t + \Delta t M$, with the displacement vector $\Delta u$, while the stress point (image of point $M$) moves in the stress space from $tP$ to $t + \Delta t P$, with the stress increment $\Delta \sigma$. Now we use the so-called return mapping approach and formulate the computational procedure according to the governing parameter method.
we have plastic deformation in time step and proceed to the plasticity calculations as follows.

The stress $t + \Delta t \sigma$, with plastic flow in the time step, is

$$t + \Delta t \sigma = t + \Delta t \sigma^E - C^E \Delta e^P$$

and must satisfy the yield condition (7). We therefore have to correct the elastic solution $t + \Delta t \sigma^E$ which corresponds to point $t + \Delta t P^E$ in Fig. 2 in order to satisfy (7); hence, we are seeking the stress point $t + \Delta t P$ on the yield surface where $t + \Delta t f_y = 0$. This procedure is generally called “return mapping” or “elastic predictor - plastic corrector” method.

Using the Euler backward method, we determine the increment of internal variables $\Delta \beta$ in the time step as

$$\Delta \beta = -\Delta \lambda C^P \frac{\partial t + \Delta t f_y}{\partial t + \Delta t \beta}$$

where, for simplicity, we assume that $C^P$ is constant in the time step.

We now express the increments of plastic strain $\Delta e^P_{ij}$ as

$$\Delta e^P_{ij} = \| \Delta e^P \| t + \Delta t n_{ij}$$

where

$$\| \Delta e^P \| = (\Delta e^P_{ij} \Delta e^P_{ij})^{1/2}$$

and $t + \Delta t n$ is the unit normal to the yield surface $t + \Delta t f_y = 0$,

$$t + \Delta t n = \frac{\partial t + \Delta t f_y / \partial t + \Delta t \sigma}{\| \partial t + \Delta t f_y / \partial t + \Delta t \sigma \|} = \frac{t + \Delta t f_y, \sigma}{\| t + \Delta t f_y, \sigma \|}$$
We further use the notation $f_{y,\sigma} \equiv \partial f_y/\partial \sigma$ for simpler writing, and the one index vector notation (e.g., $n_1 = n_{11}, \ldots, n_6 = n_{31}$). The scalar $\Delta \lambda$ in (6) and (13) can be expressed as

\begin{equation}
\Delta \lambda = \frac{\| \Delta e^P \|}{\| t+\Delta t f_{y,\sigma} \|}
\end{equation}

This relation is applicable to general plasticity models. In case of non-associated plasticity, the relations (17), (16) and (13) have a similar form, with a plastic potential function $t+\Delta t Q$ instead of $t+\Delta t f_y$.

Selecting $\| \Delta e^P \|$ as the governing parameter in Table 1, we find that

\begin{equation}
t+\Delta t \sigma = t+\Delta t \sigma^E - \| \Delta e^P \| C^E t+\Delta t \hat{n}
\end{equation}

\begin{equation}
t+\Delta t \beta = t \beta - \| \Delta e^P \| \hat{C}^P t+\Delta t \hat{n}_\beta
\end{equation}

where

\begin{equation}
\hat{C}^P = \frac{\| t+\Delta t f_{y,\beta} \|}{\| t+\Delta t f_{y,\sigma} \|} C^P
\end{equation}

t$+\Delta t \hat{n}$ is equal to $t+\Delta t n$ but contains double the shear terms because the engineering shear strains are usually used in the elastic constitutive law; and $t+\Delta t n_\beta = t+\Delta t f_{y,\beta}/\| t+\Delta t f_{y,\beta} \|$, with $t+\Delta t f_{y,\beta} \equiv \partial t+\Delta t f_y/\partial t+\Delta t \beta$. Now we can form a function $f(\| \Delta e^P \|)$ by substituting $\sigma$ and $\beta$ for a given $\| \Delta e^P \|$ into the yield function (7),

\begin{equation}
(21) f(\| \Delta e^P \|) = t+\Delta t f_y \left( t+\Delta t \sigma^E - \| \Delta e^P \| C^E t+\Delta t \hat{n}, \ t \beta - \| \Delta e^P \| \hat{C}^P t+\Delta t \hat{n}_\beta \right)
\end{equation}

where $t+\Delta t \hat{n}$ and $t+\Delta t \hat{n}_\beta$ correspond to $\| \Delta e^P \|$. It can be shown that derivative of this function is negative. Hence, the governing function $f(\| \Delta e^P \|)$ is a monotonically decreasing function, as schematically shown in Fig. 3a. This property of the function $f(\| \Delta e^P \|)$ is very important for the practical implementation of a robust computational procedure. Namely, starting with $\| \Delta e^P \| = 0$ we have $f = t+\Delta t f^E > 0$; then taking a large value $\| \Delta e^P \|_{\text{minus}}$ to obtain $f < 0$ (point P$\text{minus}$ in Fig. 3a), we can proceed by bisection or a secant method to calculate $\| \Delta e^P \|$ for which (within a tolerance) $f = 0$. The computational steps are given in Table 2.

The next task is to determine the consistent elastic-plastic tangent matrix. Using (4), (5), (12), (13) and (14) we obtain

\begin{equation}
t+\Delta t C^{EP} = t+\Delta t C^E - \Delta \sigma' \left[ \frac{\partial(\| \Delta e^P \|)}{\partial t+\Delta t e} \right]^T
\end{equation}

where $\Delta \sigma' = \partial(\Delta \sigma)/\partial(\| \Delta e^P \|) = -t+\Delta t \sigma'$, with $\Delta \sigma = t+\Delta t \sigma^E - t+\Delta t \sigma$. By differentiation of (7) with respect to $t+\Delta t e$ and solving for $\partial(\| \Delta e^P \|)/\partial t+\Delta t e$, we
Figure 3. Return mapping according to the governing parameter method. a Dependence of the governing function on increment of plastic strain $||\Delta e^P||$; b Schematic representation of search for the final stress point $t+\Delta t_P$

obtain

(23) \[ \frac{\partial (||\Delta e^P||)}{\partial t+\Delta t} = \frac{1}{a_\sigma} C^E t+\Delta t f_{y,\sigma} \]

where

(24) \[ a_\sigma = t+\Delta t f_T y_{y,\sigma} \Delta \sigma' - t+\Delta t f_T y_{y,\beta} t+\Delta t \beta' \]

and $t+\Delta t \beta' = \partial t+\Delta t \beta / \partial (||\Delta e^P||)$. In the practical implementation of (22) we can determine the derivatives numerically during the solution of the nonlinear equation (7).
1. Known quantities: \( t, \sigma, t, e, t, P, t, \beta, t, e + \Delta t e \)

2. Calculate elastic predictor \((k = 0)\)
   \[ t + \Delta t e^E = C^E \left( t + \Delta t e - t, e^P \right) \]
   If \( t + \Delta t e^E \leq 0 \) solution is elastic, \( t + \Delta t \sigma = t + \Delta t e^E; \) EXIT
   If \( t + \Delta t e^E > 0 \) plastic flow occurs in time step \( \Delta t \); hence proceed with the plasticity calculations
   \[ t + \Delta t n^{(0)} = t + \Delta t f_y^E || t + \Delta t e^E || \]
   \[ t + \Delta t n^{(0)}_\beta = t + \Delta t f_y^E || t + \Delta t e^E || \]

3. Iterations on \( ||\Delta e^P|| \) \((k = k + 1)\)
   Select \( ||\Delta e^P||^{(k)} \)
   \[ t + \Delta t e^{P(k)} = t + \Delta t e^{P(k-1)} + \left( ||\Delta e^P||^{(k)} - ||\Delta e^P||^{(k-1)} \right) t + \Delta t n^{(k-1)} \]
   Calculate stresses and internal variables
   \[ t + \Delta t \sigma^{(k)} = t + \Delta t \sigma^{(k-1)} - \left( ||\Delta e^P||^{(k)} - ||\Delta e^P||^{(k-1)} \right) C^E t + \Delta t n^{(k-1)} \]
   \[ t + \Delta t \beta^{(k)} = t + \Delta t \beta^{(k-1)} - \left( ||\Delta e^P||^{(k)} - ||\Delta e^P||^{(k-1)} \right) C^P(k-1) t + \Delta t n^{(k-1)}_\beta \]
   Check for convergence
   \[ t + \Delta t f_y^{(k)} \leq \varepsilon_f; \quad ||\Delta e^P||^{(k)} - ||\Delta e^P||^{(k-1)} || \leq \varepsilon_\Delta \]
   If convergence is reached go to step 4; otherwise calculate
   \[ t + \Delta t f_y^{(k)}, t + \Delta t f_y^{(k)}, t + \Delta t n^{(k)}_\sigma, t + \Delta t n^{(k)}_\beta, C^{P(k)} \] and go to start of step 3

4. Consistent tangent elastic-plastic matrix
   \[ t + \Delta t C^{EP} = t + \Delta t C^E - \Delta \sigma' \left[ \frac{\partial ||\Delta e^P||}{\partial t + \Delta t e} \right]^T \]

**Table 2.** Computational procedure for stress integration in plasticity according to the governing parameter method

4. Extension to Large Strains

We outline here the extension of the GPM procedure of Section 2 to large strains. A more detailed description of application of the GPM to large strain conditions is given in Kojić (2002a), and Kojić and Bathe (2004).
Basically, the difference between the stress integration in case of small strain and large strain assumptions consists in the kinematics of deformation. We briefly summarize this difference.

First, we calculate the trial elastic solution for stress $\tau^{t+\Delta t}E$ according to (8), but now the trial elastic strain $\varepsilon^{t+\Delta t}$ is determined as follows. The basic assumption in the large strain deformation is that the total deformation gradient $t_0X$ is expressed using the multiplicative decomposition (Lee and Liu 1967), as

$$t_0X = t_0X^E t_0X^P$$

where $t_0X^E$ and $t_0X^P$ are the elastic and plastic deformation gradients. Then, the trial elastic deformation gradient at end of time step $t+\Delta t_0X^E$ is calculated from

$$t+\Delta t_0X^E = t+\Delta t_0X^E t+\Delta t_0X^P$$

where $t+\Delta t_0X$ is known from the displacement field, and $t_0X^P$ is known from the history of deformation. Using the left basis concept (Simo and Hughes 1998) we further determine the trial elastic left Cauchy-Green deformation tensor,

$$t+\Delta t_0B^E = t+\Delta t_0X^E t+\Delta t_0X^P$$

where $t+\Delta t_0X$ is the relative deformation gradient. Use of the right basis is described in Bathe (1996) and Kojić and Bathe (2004). The trial elastic logarithmic strain is calculated with use of the polar decomposition theorem,

$$t+\Delta t_0e^E = \sum_i \ln \left( t+\Delta t_0\lambda^E_i \right) t+\Delta t_0\mathbf{p}^E_i \otimes t+\Delta t_0\mathbf{p}^E_i$$

where $t+\Delta t_0\lambda^E_i$ and $t+\Delta t_0\mathbf{p}^E_i$ are the principal stretches and principal vectors of $t+\Delta t_0B^E_i$. This trial elastic strain is employed in (8).

Next, the stress calculation is continued as for small strain conditions. The obtained stresses are the true (Cauchy) stresses. The computational steps are summarized in Table 3, where the expression for updating of the $t+\Delta t_0B^E_i$ is also given.

Application of the above procedure to various material models is given in, e.g., Kojić et al. (1995c), (2002b), and to shell deformation in Kojić (2002c).

5. Examples

We give two example solutions as illustration of the GPM application. The solutions are obtained using the finite element program PAK (Kojić et al. 2000) which has been continuously developing in last 25 years at University of Kragujevac. All material models and the methodology briefly presented here, are incorporated into the program PAK. The program has been extended and currently contains PAK-S for solids and structures, PAK-F for fluid mechanics, PAK-T for heat conduction, PAK-P for seepage problems, PAK-CT for coupled problems, and PAK-B for biomechanics.
1. Trial elastic state
\[ t^{+\Delta t} B^E = t^{+\Delta t} X \cdot t^{+\Delta t} Y^T \]
\[ t^{+\Delta t} e^E = \sum_i \ln (t^{+\Delta t} \lambda_i) \cdot t^{+\Delta t} \bar{p}_i^E \otimes t^{+\Delta t} \bar{p}_i^E \]
- Calculate trial elastic stress \( t^{+\Delta t} \sigma^E \) from elastic constitutive law

2. Stress integration
- Use of the appropriate material model
- Application of the governing equations for the small strain conditions
- Solution for the Cauchy stresses \( t^{+\Delta t} \sigma \)

3. Updating of variables for next time step
- Calculate the left Cauchy-Green deformation tensor
\[ t^{+\Delta t} B^E = \sum_A \exp(2 t^{+\Delta t} e_A^E) t^{+\Delta t} \bar{p}_A^E \otimes t^{+\Delta t} \bar{p}_A^E \]

Table 3. Stress integration in case of large strain elastic-plastic deformation in isotropic plasticity, use of the left basis

Example 1. Tension of a Circular Bar (Gurson Material Model). A standard smooth tensile circular specimen with the dimensions given in Fig. 4a is used to characterize the material behavior and to identify the critical damage parameters for ductile tearing at room temperature, Brocks (1995). The material is assumed to be given by the Gurson material model for which the yield condition is given as
\[ f_y = \frac{1}{2} \mathbf{S} \cdot \mathbf{S} + \frac{1}{3} \left[ 2 f^* q_1 \cosh \left( \frac{3 q_2 \sigma_m}{2 \sigma_y} \right) - 1 \right] \sigma_y^2 = 0 \]
where \( \sigma_y \) is the yield stress, \( \mathbf{S} \) and \( \sigma_m \) are the stress deviator and mean stress, \( f^* \) is a function of porosity (volume fraction) \( f \); and \( q_1, q_2 \) and \( q_3 \) are material constants. The function \( f^* \) is given as
\[ f^* = \begin{cases} f, & \text{for } f \leq f_c \\ f_c + K_f (f - f_c), & \text{for } f > f_c \end{cases} \]
where \( f_c \) is a critical porosity when the onset of rapid volume coalescence begins, and
\[ K_f = \frac{1/q_1 - f_c}{f_f - f_c} \]
Figure 4. Tension of a circular bar (Gurson material model).  

a Geometry of the specimen and boundary conditions;  
b Axial force – elongation dependence;  
c Axial force – change of diameter dependence

Here $f_f$ is the value of $f$ at the material failure. The detailed description of the model and the stress integration procedure is given in Kojić et al. (2002b).

The elastic material constants are taken as follows: Young’s modulus $E = 250$ Gpa, Poisson’s ratio $\nu = 0.3$. The uniaxial yield curve is

\begin{equation}
\sigma_y = 468.5 + 445.4 (\bar{\varepsilon}^P)^{0.361}
\end{equation}

Material constants of the model are: $q_1 = 1.5$, $q_2 = 1.0$, $q_3 = 1.5$. The initial porosity $f_0 = 0.002$, the failure porosity $f_f = 0.315$, the critical porosity $f_c = 0.05$

One half of the bar is modeled due to symmetry with the boundary conditions shown in the figure. Two-dimensional 8-node axisymmetric elements (168 elements) are used and the loading is applied through the prescribed displacements (for details of the problem modeling see Kojić et al. 2002b). The problem is solved assuming large strain conditions.
Figure 5. Compression and shear of Cam-clay material. a Geometry, boundary conditions and material data; b Prescribed displacements; c Solutions for the hardening parameter $p_0$ and mean stress $\sigma_m$; d Change of the volumetric plastic strain; e Change of the stress components; f Stress path and the yield surfaces

The final end displacement of 3.625 mm (half of the total specimen elongation) is reached in 29 equal load steps. The force-elongation and force-change of diameter
relations are shown in Figs. 4b,c. The experimental results reported by Brocks (1995) are also shown in the figures.

**Example 2. Compression and Shear of Cam-Clay Material.** Consider the large strain deformation of the Cam-clay material subjected to compression and shear. The yield condition for this model is

\[
 f_y = \sigma_m(\sigma_m - p_0) + \frac{3J_{2D}}{M} = 0
\]

where \( J_{2D} = 0.5S_{ij}S_{ij} \) is the second invariant of the stress deviator, which represents a measure for the shear loading of the material; \( p_0 \) is the horizontal axis of the ellipse (Fig. 5f) which also is the hardening function (the hardening parameter is the volumetric plastic strain); and \( M \) is the material parameter. As it is usual in geomechanics, compressive stresses and strains are considered positive.

Figure 5b shows the prescribed displacements in terms of time and material data.

One plane strain finite element, with the boundary conditions shown in Fig. 5a, is employed under the assumption of homogeneous large strain deformation. The solution is obtained using 20 equal steps (details about the solution procedure and computational model are given Kojić et al. 1995c).

Figure 5c shows the results for the hardening parameter \( p_0 \) and the mean stress \( \sigma_m \), while Figs. 5d,e show the results for the change of volumetric plastic strain and for the in-plane stress components, respectively. In Fig. 5f we show the stress path \( OC \), and the initial and final yield surfaces (with the points \( A_0 \) and \( A_1 \) on the \( \sigma_m \) axis). Note that as the shear deformation progresses, the stress state approaches the critical state \( C \), after which we have a perfectly plastic response and no volume change.

5. Concluding Remarks

We have presented the basic concept of the governing parameter method as a general implicit integration procedure for stress integration of inelastic constitutive relations within an incremental analysis of strain-driven problems. The GPM is a robust, implicit and accurate algorithm. It is applicable to all engineering conditions of material deformation (2D, 3D, shell, membrane and pipe conditions). The procedure, for which some details are given for isotropic and isothermal plasticity constitutive relations, is also applicable to other material models, such as:

- thermoplastic models
- thermoplastic and creep models
- viscoplastic models

A detailed presentation of the fundamentals of inelastic material behavior, and the GPM and its application to various material models, including large strains and numerous solved examples, are given in Kojić and Bathe (2004).
References


