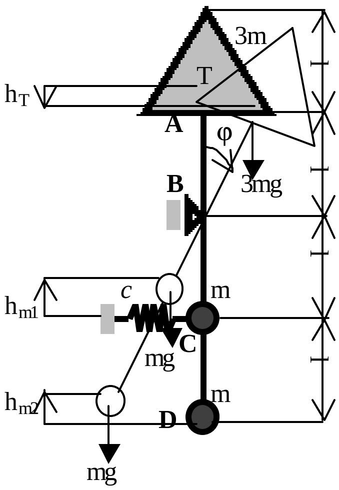
Re{ewa zadataka iz ispitnog roka januara 2001

1.zadatak



Kako je zadato zadatkom da je visina jednakostrani~nog trougla l to sledi da je stranica jednakostrani~nog trougla $a=\frac{2l\sqrt{3}}{3}$; sa slike se vidi da je sabijanje opruge: $\Delta x = l\cdot \boldsymbol{j}$, gde je ugaoni otklon φ meren od vertikale izabran za generalisanu koordinatu. Sistem ima jedan stepen slobode oscilovawa.

Potencijalna energija sistema je:

$$E_p = E_{p\Delta} + E_{pc} + E_{pm_1} + E_{pm_2} =$$

$$E_p = -2mgl\boldsymbol{j}^2 + \frac{1}{2}cl^2\boldsymbol{j}^2 + \frac{1}{2}mgl\boldsymbol{j}^2 + mgl\boldsymbol{j}^2$$

$$\Rightarrow \mathbf{E_p} = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{l}\boldsymbol{j}^2(\mathbf{cl}-\mathbf{mg})$$

Gde su: $E_{p\Delta} = 3mgh_T = 4mgl(1-\cos\boldsymbol{j}) = 2mgl\boldsymbol{j}^2$, potencijalna energija trougaone plo~ice koja je negativna, jer se vr{i pozitivan rad $+3mgh_T$ spu{tanjem centra mase plo~ice;

$E_{pc} = \frac{1}{2}c\Delta x^2 = \frac{1}{2}cl^2\boldsymbol{j}^2$, potencijalna energija opruge, jednaka deformacionom radu koji je izvr{en na sabijawu opruge za $\Delta x = l\cdot\boldsymbol{j}$;

$E_{pm_1} = mgh_{m_1} = mgl(1-\cos\boldsymbol{j}) = \frac{1}{2}mgl\boldsymbol{j}^2$, potencijalna energija mase m u ta~ki C koja se podi`e za h_{m1};

$E_{pm_2} = mgh_{m_2} = mg2l(1-\cos\boldsymbol{j}) = mgl\boldsymbol{j}^2$, potencijalna energija mase m u ta~ki D koja se podi`e za h_{m2}.

U polo`aju ravnote`e sisitema, potencijalna energija ima ekstrmnu vrednost pa je prvi izvod E_p po generalisanoj koordinati jednak nuli : $\frac{\partial E_p}{\partial \boldsymbol{j}} = l\boldsymbol{j}(cl-mg) = 0$ odakle sledi veza parametara sisitema :

$$cl = mg .$$

Uslov da polo`aj ravnote`e bude stabilan je da E_p bude u minimumu, pa je potrebno da je:

$$\frac{\partial^2 E_p}{\partial \boldsymbol{j}^2} \succ 0 \Rightarrow l(lc-mg) \succ 0 \Rightarrow \qquad c \succ \frac{mg}{l}:$$

Kineti~ka energija sistema je: $E_k = E_{k\Delta} + E_{km_1} + E_{km_2} \Rightarrow \mathbf{E_k} = \frac{\mathbf{32}}{\mathbf{6}}\mathbf{ml}^2\dot{\boldsymbol{j}}^2$;

Gde su:

$E_{k\Delta} = \frac{1}{2}J_{BT}\dot{\boldsymbol{j}}^2 = \frac{17}{6}ml^2\dot{\boldsymbol{j}}^2$; kineti~ka energija trougaone plo~ice, gde je J_{VT} aksijalni moment inrcjie mase plo~ice za osu obrtawa kroz V:

$$J_{BT} = J_T + \overline{BT}^2 3m = \frac{1}{3}ml^2 + \left(\frac{4}{3}l\right)^2 3m = \frac{17}{3}ml^2,$$

$$J_T = \boldsymbol{r} I_T = \frac{M}{A}\left(I_x + I_h\right) = \frac{3m}{\frac{l^2\sqrt{3}}{3}} 2\frac{a^4\sqrt{3}}{96} = \frac{1}{3}ml^2,$$

$E_{km1} = \frac{1}{2}m v_{m1}^2 = \frac{1}{2}ml^2 \dot{\boldsymbol{j}}^2$, kineti~ka energija mase m u ta~ki C ,

$E_{km2} = \frac{1}{2}m v_{m2}^2 = \frac{1}{2}m4l^2 \dot{\boldsymbol{j}}^2 = 2ml^2 \dot{\boldsymbol{j}}^2$, kineti~ka energija mase m u ta~ki D.

Drugi na~in: $E_k = \frac{1}{2}J_B \dot{\boldsymbol{j}}^2$, gde je:

$$J_B = J_{BT} + ml^2 + m(2l)^2 = \frac{32}{3}ml^2,$$

Aksijalni moment inercije mase sisterma za osu kroz V.

Lagrang-eova jedna~ina druge vrste za generalisanu koordinatu φ je:

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{\boldsymbol{j}}} - \frac{\partial E_k}{\partial \boldsymbol{j}} + \frac{\partial E_p}{\partial \boldsymbol{j}} = 0 \Rightarrow \frac{32}{3}ml^2 \ddot{\boldsymbol{j}} + l\boldsymbol{j}(cl - mg) = 0$$

$$\ddot{\boldsymbol{j}} + \boldsymbol{w}^2 \boldsymbol{j} = 0$$

Odavde je kvadrat kru`ne frekvencije malih oscilacija oko polo`aja stabilne ravnote`e:

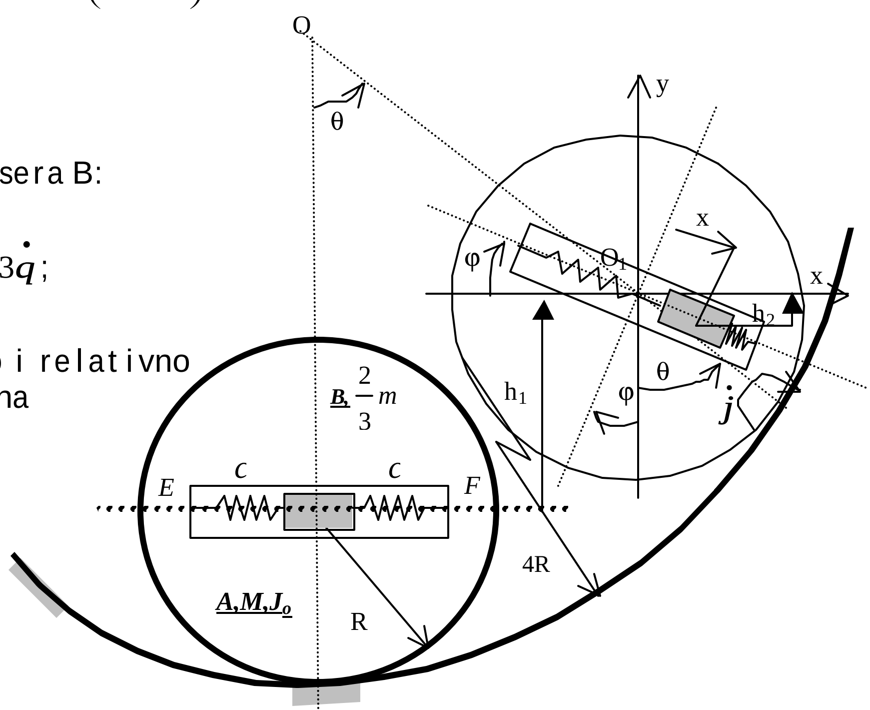
$$\boldsymbol{v}^2 = \frac{\mathbf{3}}{\mathbf{32}}\left(\frac{\mathbf{c}}{\mathbf{m}} - \frac{\mathbf{g}}{\mathbf{l}}\right)$$

2. zadatak

Odre|ivanje brzine balansera B:

$$\overline{OO_1} = 3R; \quad v_{01} = R\dot{\boldsymbol{j}} \qquad \Rightarrow \dot{\boldsymbol{j}} = 3\dot{\boldsymbol{q}};$$
$$v_{01} = 3R\dot{\boldsymbol{q}}$$

Balanser izvodi prenosno i relativno kretawe pa je wegova brzina



$\mathbf{v}_B = \vec{v}_r + \vec{v}_p$

$v_r = \dot{x}; \quad \vec{v}_p = \vec{v}_{01} + \vec{v}_B^{\,01};$

$v_{Bx} = \dot{x} + v_{01}\cos(j + q) = \dot{x} + 3R\dot{q}\cos 4q$

$v_{By} = v_{01}\sin(j + q) - v_B^{01} = 3R\dot{q}\sin 4q - 3Rx\dot{q}$

$v_B^2 = v_{Bx}^2 + v_{By}^2 = \left(\dot{x} + 3R\dot{q}\cos 4q\right)^2 + \left(3R\dot{q}\sin 4q - 3\dot{x}R\dot{q}\right)^2$

$= \dot{x}^2 + \left(3R\dot{q}\right)^2 + 6R\dot{q}\dot{x}\cos 4q - 9R^2\dot{q}^2x^2\sin 4q$

^etvrti ~lan ove jedna~ine zanemarujemo kao malu veli~inu vi{eg reda $e^3 \approx -x^2\dot{q}^2 \approx 0$ i
posle razvijawa funkcija $\sin 4q$ n $\cos 4q$ u Taylor-ov red i zanemarivawa kvadratnog i ostalih ~lanova kao malih veli~ina vi{eg red $\sin 4q \approx 4q, \quad \cos 4q \approx 1$ sledi:

$$v_B^2 \approx \left(\dot{x} + 3R\dot{q}\right)^2.$$

Kineti~ka energija sistema: $E_k = E_{kA} + E_{kB} = 24mR\dot{q}^2 + 2mR\dot{x}\dot{q} + \frac{1}{3}m\dot{x}^2 \Rightarrow \mathbf{A} = \begin{pmatrix} 48mR^2 & 2mR \\ 2mR & \frac{2}{3}m \end{pmatrix}$

Gde su: $E_{kA} = \frac{1}{2}M v_{01}^2 + \frac{1}{2}J_{01}\dot{j}^2 = \frac{1}{2}M9R^2\dot{q}^2 + \frac{1}{2}J_{01}9\dot{q}^2$,kineti~ka energija sfere A;

$E_{kB} = \frac{1}{2}\frac{2}{3}m v_B^2 = \frac{1}{2}\frac{2}{3}m\left(x^2 + 6R\dot{x}\dot{q} + 9R^2\dot{q}^2\right)$, kineti~ka energjia balansera B.

Promena potencijalne energije sistema pri poreme}aju-izlasku iz ravnote`nog polo`aja:

$E_p = E_{pA} + E_{pB} + 2E_{pc} = \frac{3}{2}MgRq^2 + \frac{1}{3}mg\left(3Rq^2 - 6xq\right) + cx^2 =$

$= \frac{3}{2}Rq^2g\left(M + \frac{2}{3}m\right) - 2mgxq + cx^2 = 6Rgq^2 - 2mgxq + cx^2 \Rightarrow \mathbf{C} = \begin{pmatrix} 12mgR & -2mg \\ -2mg & 2c \end{pmatrix}$

Gde su: $E_{pA} = Mgh_1 = Mg3R(1 - \cos q) = \frac{3}{2}MgRq^2$, promena potencijalne energije sfere A;

$E_{pB} = \frac{2}{3}mg(h_1 - h_2) = \frac{2}{3}mg[3R(1 - \cos q) - x\sin j] = \frac{1}{3}mg\left(3Rq^2 - 6xq\right)$, promena potencijalne energije balansera B;

$E_{pc} = 2\frac{1}{2}cx^2$, promena potencijalne energije opruga pri deformisawu.

Lagrange-eove jedna~ine druge vrste za generalisane koordinate θ i x, u matri~nom obliku su:

$\mathbf{A}\begin{Bmatrix}\ddot{\theta}\\ \ddot{x}\end{Bmatrix}+\mathbf{C}\begin{Bmatrix}\theta\\ \mathbf{x}\end{Bmatrix}=0$; Pretpostavimo re{ewe: $\begin{matrix}\theta = A_1\cos(\omega t+\alpha); & \ddot{\theta}=-\omega^2 A\,\theta\\ x = A_2\cos(\omega t+\alpha); & \ddot{x}=-\omega^2 A_2 x\end{matrix}$;

pa iz sisitema Lagrange- ovih jedna~ina dobijamo sisitem homogenih algebarskih jedna~ina ~iji je matri~ni oblik po nepoznatim amplitudama A_1 i A_2:

$\left|-\omega^2\mathbf{A}+\mathbf{C}\right|\begin{Bmatrix}A_1\\ A_2\end{Bmatrix}=0$. Odavde se dobija frekventna jedna~ina iz uslova da je determinanta sistema homogenih algebarskih jedna~ina jednaka nuli:

$$f\left(\omega^2\right)=\left|\mathbf{C}-\omega^2\mathbf{A}\right|=0$$

$f\left(\omega^2\right)=\begin{vmatrix}6Rg-24\omega^2R^2 & -g-R\omega^2\\ -g-R\omega^2 & -\frac{1}{3}\omega^2+\frac{c}{m}\end{vmatrix}=0$; Posle uvo|ewa zadatih odnosa ra~unamo vrednost determinante:

$$R^2\left(-2k\omega^2+8\omega^4+k^2-4k\omega^2\right)-R^2\left(k^2+2k\omega^2+\omega^4\right)=0;\ k=\frac{g}{R};\frac{c}{m}=\frac{1}{6}k;$$

$$-8k\omega^2+7\omega^4=0\Rightarrow\omega^2\left(7\omega^2-8k\right)=0$$

Koreni frekventne jedna~ine su sopstvene kru`ne frekvencije sisitema koje iznose:

$$\omega_1^2=0;$$

$$\omega_2^2=\frac{8}{7}k.$$

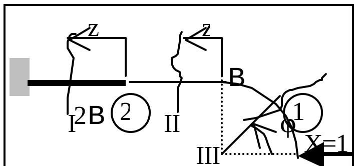
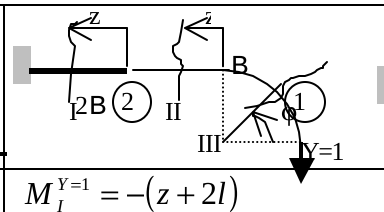
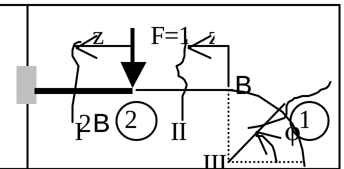
Dakle, sistem ima dva stepena slobode kretawa , ali jedan stepen slobode oscilovawa, za zadate parametre sistema.

DRUGI NA^IN: Ako se za generalisane koordinate izaberu: $R\theta,\quad x$

$$C=\frac{2mg}{R}\begin{pmatrix}6 & -1\\ -1 & k\end{pmatrix}\quad A=\frac{2}{3}m\begin{pmatrix}72 & 3\\ 3 & 1\end{pmatrix}\quad f\left(u=\frac{R\omega^2}{3g}\right)=\begin{vmatrix}6-72u & -1-3u\\ -1-3u & k-u\end{vmatrix}=0\quad f(u)=3u(8-21u)=0$$

$$u_1=0,\ u_2=\frac{8}{21};\quad \omega_1^2=0;\quad \omega_2^2=\frac{8g}{7R}$$

3.zadatak

				
Presek I, *0<z<l,2B*	dz	$M_I^{x=1}=-l$	$M_I^{Y=1}=-(z+2l)$	$M_I^{F=1}=-z$
Presek II, *0<z<l,B*	dz	$M_{II}^{x=1}=-l$	$M_{II}^{Y=1}=-(z+l)$	$M_{II}^{F=1}=0$
Presek III, *$0<\varphi<\pi/2$,B*	ldφ	$M_{III}^{x=1}=-l\sin\varphi$	$M_{III}^{Y=1}=-l(1-\cos\varphi)$	$M_{III}^{F=1}=0$

$$\boldsymbol{a}_{22}^{HH} = \frac{1}{2\boldsymbol{b}}\int_0^l l^2 dz + \frac{1}{\boldsymbol{b}}\int_0^l l^2 dz + \frac{l^3}{\boldsymbol{b}}\int_0^{p/2} \sin^2 \boldsymbol{j} d\boldsymbol{j} = \frac{l^3}{2\boldsymbol{b}} + \frac{l^3}{2\boldsymbol{b}}\int_0^{p/2}(1-\cos 2\boldsymbol{j})d\boldsymbol{j} = \frac{l^3}{2\boldsymbol{b}} + \frac{l^3}{\boldsymbol{b}} + \frac{\boldsymbol{p}l^3}{4\boldsymbol{b}}$$

$$\boldsymbol{a}_{22}^{HH} = \frac{l^3}{4\boldsymbol{b}}(6+\boldsymbol{p}) = p(6+\boldsymbol{p})$$

$$\boldsymbol{a}_{22}^{HV} = \boldsymbol{a}_{22}^{VH} = \frac{l}{2\boldsymbol{b}}\int_0^l (z+2l)dz + \frac{l}{\boldsymbol{b}}\int_0^l (z+l)dz + \frac{l^3}{\boldsymbol{b}}\int_0^{p/2}(1-\cos\boldsymbol{j})\sin\boldsymbol{j}d\boldsymbol{j} = \frac{5l^3}{4\boldsymbol{b}} + \frac{3l^3}{2\boldsymbol{b}} + \frac{l^3}{2\boldsymbol{b}} = \frac{13l^3}{4\boldsymbol{b}}$$

$$\boldsymbol{a}_{22}^{HV} = \boldsymbol{a}_{22}^{VH} = 13p$$

$$\boldsymbol{a}_{22}^{VV} = \frac{1}{2\boldsymbol{b}}\int_0^l (z+2l)^2 dz + \frac{1}{\boldsymbol{b}}\int_0^l (z+l)^2 dz + \frac{l^3}{\boldsymbol{b}}\int_0^{p/2}(1-\cos\boldsymbol{j})^2 d\boldsymbol{j} = \frac{l^3}{2\boldsymbol{b}}\left(\frac{1}{3}+6\right) + \frac{l^3}{\boldsymbol{b}}\left(\frac{1}{3}+2\right) + \frac{l^3}{\boldsymbol{b}}\left(2+\frac{3\boldsymbol{p}}{4}\right)$$

$$\boldsymbol{a}_{22}^{VV} = \frac{l^3}{4\boldsymbol{b}}(14+3\boldsymbol{p}) = p(14+3\boldsymbol{p})$$

$$\boldsymbol{a}_{21}^{HV} = \frac{l}{2\boldsymbol{b}}\int_0^l z dz + 0 = \frac{l^3}{4\boldsymbol{b}} = p$$

$$\boldsymbol{a}_{21}^{VV} = \frac{1}{2\boldsymbol{b}}\int_0^l z(z+2l)dz + 0 = \frac{2l^3}{3\boldsymbol{b}} = \frac{8}{3}p$$

Diferencijalne jedna~ine oscilovawa materijalne ta~ke:

$$x = \boldsymbol{a}_{22}^{HH}\left(-m\ddot{x}\right) + \boldsymbol{a}_{22}^{HV}\left(-m\ddot{y}\right)t$$

$$y = \boldsymbol{a}_{22}^{HV}\left(-m\ddot{x}\right) + \boldsymbol{a}_{22}^{VV}\left(-m\ddot{y}\right);$$

$$x = p(6+\boldsymbol{p})\left(-m\ddot{x}\right) + 13p\left(-m\ddot{y}\right);$$

$$y = 13p\left(-m\ddot{x}\right) + p(14+3\boldsymbol{p})\left(-m\ddot{y}\right);$$

$$pm(6+\boldsymbol{p})\ddot{x} + x + 13pm\ddot{y} = 0$$

$$13pm\ddot{x} + y + pm(14+3\boldsymbol{p})\ddot{y} = 0.$$

Diferencijalne jedna~ine prinudnih oscilovawa materijalne ta~ke:

$$x = \boldsymbol{a}_{22}^{HH}\left(-m\ddot{x}\right) + \boldsymbol{a}_{22}^{HV}\left(-m\ddot{y}\right) + \boldsymbol{a}_{21}^{HV}F_0\cos\boldsymbol{w}t$$

$$y = \boldsymbol{a}_{22}^{HV}\left(-m\ddot{x}\right) + \boldsymbol{a}_{22}^{VV}\left(-m\ddot{y}\right) + \boldsymbol{a}_{21}^{VV}F_0\cos\boldsymbol{w}t;$$

$$pm(6+\boldsymbol{p})\ddot{x} + x + 13pm\ddot{y} = pF_0\cos\boldsymbol{w}t.$$

$$13pm\ddot{x} + y + pm(14+3\boldsymbol{p})\ddot{y} = \frac{8}{3}pF_0\cos\boldsymbol{w}t.$$

Pretpostavimo re{ewa:

$$x = C_1\cos\Omega t;\ \ddot{x} = -\Omega^2 C_1 x$$
$$y = C_2\cos\Omega t;\ \ddot{y} = -\Omega^2 C_2 y$$

, i uvode}i oznaku $h = \frac{1}{3}pF_0$, slede sistemi homogenih algebarskih jedna~ina po nepoznatim C_1 i C_2:

$$\left[1 - pm\Omega^2(6+\boldsymbol{p})\right]C_1 - 13pm\Omega^2 C_2 = 3h,$$

$-13pm\Omega^2 C_1 + \left[1 - pm\Omega^2(14 + 3\boldsymbol{p})\right]C_2 = 8h$;

Determinanta ovog sisitema je:

$$f\left(v = pm\Omega^2\right) = \begin{vmatrix} 1-(6+\boldsymbol{p})v & -13v \\ -13v & 1-(14+3\boldsymbol{p})v \end{vmatrix} = \Delta(v).$$

Determinanta sistema treba da je razli~ita od nule da ne bi do{lo do rezonancije $\Delta(v) \neq 0$.

Da ne bi bilo oscilovawa u vertikalnm pravcu:

$$y = 0 \Rightarrow C_2 = 0,$$
$$C_2 = \frac{\Delta_{C2}}{\Delta} \Rightarrow \Delta_{C2} = 0;$$

$$\Delta_{C2} = \begin{vmatrix} 1-(6+\boldsymbol{p})v & 3h \\ -13v & 8h \end{vmatrix} = h\ \left[8 - 8v(6+\boldsymbol{p}) + 39v\right] = 0 \Rightarrow h\ \left[8 - v(9+8\boldsymbol{p})\right] = 0 \Rightarrow$$

$$v_a = \frac{8}{9+8\boldsymbol{p}} \Rightarrow$$
$$\Omega_a^2 = \frac{8}{pm(9+8\boldsymbol{p})}.$$

Kada sistem prinudno osciluje ugaonom frekvencom Ω_a tada je on za materijalnu ta~ku dinami~ki apsorber za vertikalni pravac.

U horizontalnom pravcu sisitem ne mo`e da se pona{a kao dinami~ki apsorber jer ni za jednu frekvenciju amplituda $C_1$ ne mo`e biti nula.

4.zadatak

Parcijalna diferencijalna jedna~ina torzijskih oscilacija homogenog vratila je:

$$\frac{\partial^2 \boldsymbol{q}(z,t)}{\partial t^2} = c_t^2 \frac{\partial^2 \boldsymbol{q}(z,t)}{\partial z^2}, c_t = \sqrt{\frac{G}{\boldsymbol{r}}}, \qquad (1)$$

gde je $\boldsymbol{q}(z,t)$ ugao obrtawa popre~nih preseka vratila.

Re{ewe ove jedna~ine predpostavqamo u obliku:

$\boldsymbol{q}(z,t) = T(t)Z(z)$.

Saglasno Bernoulli-jevoj metodi partikularnih integrala, parcijalna diferencijalna jedna~ina se svodi na dve obi~ne diferencijalne jedna~ine razdvojenih promenqivih:

$\ddot{T}(t) + \boldsymbol{w}^2 T(t) = 0,$ gde je : $\boldsymbol{w} = c_t\, \boldsymbol{l}$,

$Z''(z) + \boldsymbol{l}^2 Z(t) = 0,$

~ija su re{ewa za levu i desnu stranu vratila:

$Z^l(z) = C_1 \cos \boldsymbol{l}z + C_2 \sin \boldsymbol{l}z$, $\quad Z^d(z) = D_1 \cos \boldsymbol{l}z + D_2 \sin \boldsymbol{l}z$

$T^l(t) = A\cos \boldsymbol{w}t + B\sin \boldsymbol{w}t$, $\quad T^d(t) = A\cos \boldsymbol{w}t + B\sin \boldsymbol{w}t$, koja su zbog simetrije imaju jednak oblik.

Jedna~ina (1) sada postaje: $\ddot{T} + \boldsymbol{l}^2 c_t^2 T = 0 \Rightarrow \ddot{T} = -\boldsymbol{l}^2 c_t^2 T$.

Grai~ni uslovi:

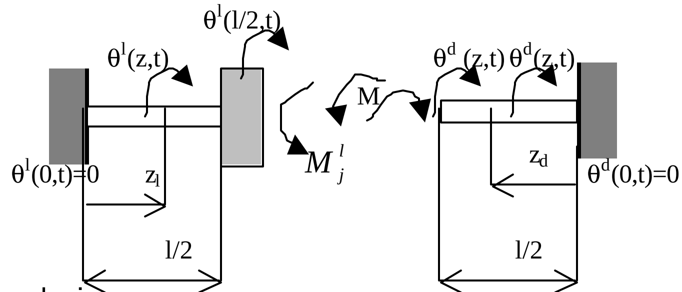
$\boldsymbol{q}^l(0,t) = 0 \Rightarrow \boldsymbol{q}(0,t) = Z(0)T(0) = 0 \Rightarrow Z_l(0) = 0 \Rightarrow C_1 = 0,$

$\boldsymbol{q}^d(0,t) = 0 \Rightarrow \boldsymbol{q}^d(0,t) = Z(0)T(0) = 0 \Rightarrow Z_d(0) = 0 \Rightarrow D_1 = 0, \Rightarrow$

$Z^l(z) = C_2 \sin \lambda z$, $\qquad Z^d(z) = D_2 \sin \lambda z$

$$Z_l = l/2 \Rightarrow GI_0 \frac{\partial \theta^l(l/2,t)}{\partial z^l} = M_j^l + M^l, \quad GI_0 \frac{\partial \theta^l(l/2,t)}{\partial z^l} = -J_z \frac{\partial^2 \theta^d(l/2,t)}{\partial t^2} + M^l \quad (2)$$

$$Z_d = l/2 \Rightarrow GI_0 \frac{\partial \theta^d(l/2,t)}{\partial z^d} = -M^d \quad (3)$$



Po{ to je $\theta^l(l/2,t) = \theta^d(l/2,t)$ i $M^l + M^d = 0$ kada saberemo jedna~ine (2) i (3) dobjemo:

$$2GI_0 \frac{\partial \theta}{\partial z} + J_z \frac{\partial^2 \theta}{\partial t^2}\bigg|_{z = l/2} = 0 .$$

Po{ to je : $\frac{\partial^2 \theta}{\partial t^2} = Z\ddot{T} = -\lambda^2 c_t^2 TZ = -\lambda^2 c_t^2 \theta(t)$, onda imamo:

$$2GI_0 Z`(l/2)T - J_z \lambda^2 c_t^2 Z(l/2)T = 0 / T \neq 0 \Rightarrow$$

$$2GI_0 \lambda C_2 \cos \lambda \frac{l}{2} - J_z \lambda^2 c_t^2 C_2 \sin \lambda \frac{l}{2} = 0; \ \lambda \cdot l = \xi \Rightarrow$$

$2GI_0 \cos \frac{\xi}{2} - J_z \lambda \frac{G}{\rho} \sin \frac{\xi}{2} = 0$, odakle sledi frekventna jedna~ina

$tg \frac{\xi}{2} = \frac{2I_0 \rho l}{J_z \xi}$, uvode} i smenu $\mu = \frac{J_z}{\rho I_0 l}$ dobijamo:

$$tg \frac{\xi}{2} = \frac{2}{\mu \xi},$$

gde postoji n korena $\xi_1, \xi_2, \xi_3 \ldots$

Ako uvedemo aproksimaciju :

$$tg \frac{\xi}{2} \approx \frac{\xi}{2} \Rightarrow$$

$\frac{\xi}{2} \approx \frac{2I_0 \rho l}{J_z \xi} \Rightarrow \xi^2 = \frac{4I_0 \rho l}{J_z}$, { to predstavqa prvu aproksimaciju re{ ewa.

Zbog uvedenih smena sledi: $\omega_n = \frac{\xi_n}{l}\sqrt{\frac{G}{\rho}}$.

Stoga sledi da je pribli`na vrednost najni`e kru`ne frekvencije:

$$\omega_1^2 = c_t^2 \frac{4I_0 \rho}{J_z l} = \frac{4GI_0}{J_z l}.$$

Koriste}i analogiju izme|u torzijskih oscilacija vratila kru`no prstenog popre~nog preseka i longitudinalnih oscilacija { tapa dobija se:

$$\frac{\partial^2 w(z,t)}{\partial t^2} = c^2 \frac{\partial^2 w(z,t)}{\partial z^2}, c^2 = \frac{E}{\rho}$$

$w(z,t) = T(t)Z(z)$.

~ija su re{ ewa za levu i desnu stranu { tapa:

$Z^l(z) = C_1 \cos \lambda z + C_2 \sin \lambda z$, $\qquad Z^d(z) = D_1 \cos \lambda z + D_2 \sin \lambda z$

$T^l(t) = A\cos wt + B\sin wt$, $T^d(t) = A\cos wt + B\sin wt$, koja su zbog simetrije jednakog oblika.

Grai~ni uslovi:

$w^l(0,t) = 0 \Rightarrow w(0,t) = Z(0)T(0) = 0 \Rightarrow Z_l(0) = 0 \Rightarrow C_1 = 0,$

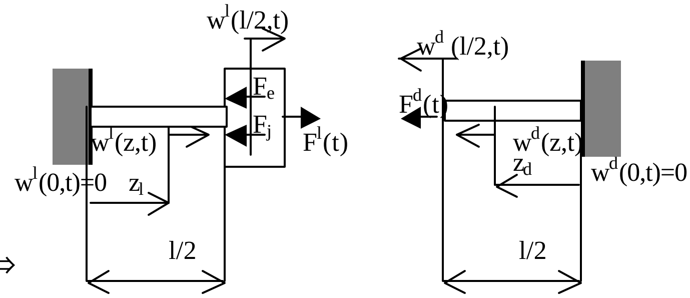
$w^d(0,t) = 0 \Rightarrow w^d(0,t) = Z(0)T(0) = 0 \Rightarrow Z_d(0) = 0 \Rightarrow D_1 = 0, \Rightarrow$

$Z^l(z) = C_2 \sin \mathbf{l} z$, $Z^d(z) = D_2 \sin \mathbf{l} z$

$Z_l = l/2 \Rightarrow F_e^l = F_j + F(t)$, $EA\dfrac{\partial w^l(l/2,t)}{\partial z^l} = -m\dfrac{\partial^2 w^d(l/2,t)}{\partial t^2} + F(t)$ (2)

$Z_d = l/2 \Rightarrow F_e^d = -F(t),\ EA\dfrac{\partial w^d(l/2,t)}{\partial z^d} = -F(t)$ (3)

Kada saberemo jedna~ine (2) i (3) dobjemo:



$2EA\dfrac{\partial w}{\partial z} + m\dfrac{\partial^2 w}{\partial t^2}\Bigg|_{z=l/2} = 0$,

$2EAZ`(l/2)T - m\mathbf{l}^2c^2Z(l/2)T = 0 / T \neq 0 \Rightarrow$

$2EA\mathbf{l}C_2\cos\mathbf{l}\dfrac{l}{2} - m\mathbf{l}^2c^2C_2\sin\mathbf{l}\dfrac{l}{2} = 0;\ \mathbf{l}\cdot l = \mathbf{x} \Rightarrow$

$2EA\cos\dfrac{\mathbf{x}}{2} - m\mathbf{l}\dfrac{E}{\mathbf{r}}\sin\dfrac{\mathbf{x}}{2} = 0$, odakle sledi frekventna jedna~ina:

$$tg\frac{\mathbf{x}}{2} = \frac{2EA}{m\mathbf{l}c^2},$$

uvode} i smenu $m = \dfrac{m}{\mathbf{r}Al}$ dobijamo:

$$tg\frac{\mathbf{x}}{2} = \frac{2}{m\mathbf{x}},$$

gde postoji n korena ξ1, ξ2, ξ3...

Ako uvedemo aproksimaciju :

$tg\dfrac{\mathbf{x}}{2} \approx \dfrac{\mathbf{x}}{2} \Rightarrow \Rightarrow \mathbf{x}^2 = \dfrac{4A\mathbf{r}l}{m}$, {to predstavqa prvu aproksimaciju re{ewa.

Zbog uvedenih smena sledi: $\mathbf{w}_m = \dfrac{\mathbf{x}_m}{l}\sqrt{\dfrac{E}{\mathbf{r}}}$

Stoga sledi da je pribli`na vrednost najni`e kru`ne frekvence:

$$\mathbf{w}_1^2 = c^2\frac{4A\mathbf{r}}{ml} = \frac{4EA}{ml}.$$

Da bi najni`e kru`ne frekfencije bile jednake potrebno je da bude ispuwen uslov:

$$\frac{I_0 G}{J_z} = \frac{EA}{m}.$$

ANALOGIJA
TORZIJSKIH (UVOJNIH) I LONGITUDINALNIH (UZDU@NIH) OSCILACIJA

Koriste}i analogiju izme|u torzijaskih oscilacija vratila i longitudinalnih oscilacija {tapa, datu u narednoj tabeli, do ovog rezultata se mo`e do}i i direktno:

Torzijske oscilacije		Longitudinalne oscilacije
$\theta(z,t)$	$\rightarrow$	$w(z,t)$
G	$\rightarrow$	E
J_z	$\rightarrow$	m
c_t	$\rightarrow$	c
I_0	$\rightarrow$	A
ρ	$\rightarrow$	ρ