# Why constructive topology must be point-free

based on j.w.w. G. Sambin

Maria Emilia Maietti

University of Padova

*Constructive Mathematics, Foundations and Practice*

*24-28 June 2013, Niš, Serbia*

# *why constructive topology must be point-free*

because

topology in the minimalist foundation must be pointfree

minimalist foundation= our proposed foundation

as a COMMON CORE

among constructive

and (classical) foundations

## *why topology in the minimalist foundation must be point-free*

the minimalist foundation is compatible with classical predicativity

($\mathcal{P}(Nat)$ is not a set!)

In particular

reals CANNOT form a set in the minimalist foundation

both in Dedekind or Cauchy form

(as well as choice sequences of Cantor or Baire space)

$\Rightarrow$ reals can be represented only in a point-free way

in the minimalist foundation!

## *in most relevant constructive foundations*

reals (both Dedekind reals and Cauchy ones) do form a set

(for example in Aczel's CZF)

if countable choice holds

$\Rightarrow$ Cauchy reals= Dedekind reals

(for example in setoid model of Martin-Löf's type theory )

# *why constructive topology must be point-free*

conceptual motivation:

point-free topology generalizes point-wise one

via Sambin's embedding of

concrete spaces (point-wise topologies)

into

positive topologies (point-free topologies with primitive closed subsets)

# *Plan of the talk*

- why building a minimalist two-level foundation for constructive mathematics
- why reals cannot form a set in the minimalist foundation
- point-free presentation of reals, in particular of Bishop reals.

why building a minimalist two-level foundation for constructive mathematics ??

## *Usefulness of constructive mathematics*

$\boxed{\text{to develop MATHEMATICS with IMPLICIT computational contents}}$

via a computational understanding of abstract mathematics

# *Usefulness of constructive mathematics*

| | | |
|---|---|---|
| constructive mathematics | conciliates | abstract mathematics |
| | | with |
| | | computational mathematics |

## To make EXPLICIT usefulness of constructive mathematics

build a foundation conciliating different kinds of foundations:

| | |
|---|---|
| abstract mathematics | in a set-theoretic foundation<br>à la Zermelo Fraenkel |
| computational mathematics | in a type theoretic foundation |

$$\Downarrow$$

build a TWO-LEVEL foundation for constructive mathematics

## *example of a two-level foundation*

Aczel's CZF (usual math language)

$\Downarrow$ (interpreted in)

Martin-Löf's type theory (reliable programming language

and intensional type theory )

## *why do we need two levels?*

is Aczel's CZF (usual math language) enough for formalizing constructive mathematics?

NO, the interpretation in INTENSIONAL type theory is crucial IF we want to make EXPLICIT how

1. to extract computational contents of CZF-proofs and formalize them in a reliable proof assistant

2. to reduce CZF-consistency to a reliable theory

## *why do we need two levels?*

underlying assumption:

we trust INTENSIONAL predicative type theory

(like Martin-Löf's type theory)

as a primitive consistent theory

$\Longrightarrow$ consistency of other needed foundational theories

(with a better language close to matematical practice)

can be reduced to such a basic theory

## *our IDEAL constructive foundation as a TWO-LEVEL theory*

1. an intensional + predicative + constructive level

(with decidable equality of sets and elements)

SUITABLE as a RELIABLE base for a proof-assistant

\+

2. an extensional level to represent abstraction/quotients

(with undecidable equality of sets and elements)

## *our IDEAL constructive foundation as a TWO-LEVEL theory*

1. an intensional + predicative + constructive level

(with decidable equality of sets and elements)

SUITABLE as a RELIABLE base for a proof-assistant

+

2. an extensional level to represent abstraction/quotients

(with undecidable equality of sets and elements)

+ MINIMALITY

NOTE: the two levels are IRREDUCIBLE

## *Plurality of constructive foundations* $\Rightarrow$ *need of a minimalist foundation*

| | classical | constructive |
|---|---|---|
| | ONE standard | NO standard |
| impredicative | Zermelo-Fraenkel set theory | internal theory of topoi<br>Coquand's Calculus of Constructions |
| predicative | Feferman's explicit maths | Aczel's CZF<br>Feferman's constructive expl. maths<br>Martin-Löf's type theory |

what common core ??

Aczel’s CZF is not a minimalist theory!

## *why CZF is not minimalist*

Aczel's CZF (usual math language)

$\Downarrow$ (interpreted in)

Martin-Löf's type theory (reliable programming language)

1. use of choice principles is relevant to interpret CZF axioms in type theory

2. CZF + EM= ZF

   $\Rightarrow$ CZF is NOT compatible with CLASSICAL PREDICATIVITY

   (such as with Feferman's theories)

## *our GOAL: build a MINIMALIST foundation*

we want to build a MINIMALIST two-level foundation

for constructive mathematics

to view other foundations as extensions in a MODULAR way

(for MODULAR REUSE of PROOFS)

at the right level (intensional or extensional)

## *HOW to build a MINIMALIST foundation*

Aczel's CZF (usual math language)

`WITHOUT universes of sets`

`+ local membership`

`+ add typed terms` $\neq$ `functional relations`

$\Downarrow$ (interpreted in)

Martin-Löf's type theory

`+ propositions added primitively`

as Aczel-Gambino logic enriched type theory [JSL 2006]

`but with proof-terms`

`no choice principles, no unique choice`

## desired link between levels in the minimalist foundation

| we require the link between levels to be local and modular |
|---|

$\Rightarrow$ NO use of choice principles to interpret the extensional level

$\Rightarrow$ NO extensional equality of sets BUT OF subsets

since our EXTENSIONAL THEORY is a LOCAL SET THEORY

(= set membership is localized to a given primitive set)

# Our two level minimalist constructive foundation

from [M.-Sambin'05],[M.'09]

| | | |
|---|---|---|
| emTT | = | extensional minimalist level |
| $\Downarrow \mathcal{I}$ | | (interpretation via quotient completion) |
| mtt | = | intensional minimalist type theory |
| | | = `predicative CoC` |

emtt $\Rightarrow$ interpretable in $\begin{cases} \text{Aczel's CZF} \\ \text{predicative classical set theory} \end{cases}$

mtt is a fragment of Martin-Löf's type theory (with only one universe) after identifying props-as-sets.

[M.'09] "A minimalist two-level foundation for constructive mathematic", 2009

[M.-Sambin'05] "Toward a minimalist foundation for constructive mathematics", 2005

## *the two-level foundation needs an extra level!*

Because of strong link between intensional/extensional levels:

| | |
|---|---|
| two-level foundation | $\left\{ \begin{array}{l} \text{extensional level} \\ \text{intensional level} \end{array} \right.$ |
| for computer extraction | realizability level |

intensional level $\neq$ realizability level

for minimality of the extensional level!

for ex: “all number theoretic functions are recursive” holds

at the realizability level

but canNOT be lifted at the extensional level

for compatibility with classical extensional levels

## *our PREDICATIVE theory needs TWO SIZE ENTITIES*

a predicative theory = theory with NO IMPREDICATIVE constructions

$\Longrightarrow$ for ex. power of subsets is a COLLECTION NOT a set

$$\Downarrow$$

predicative set theory makes essential use of 2 sizes:

SETS + COLLECTIONS

## *in our minimalist foundation*

sets= inductively generated entities + induction principle

collection=entities with no induction principle

$$\Downarrow$$

there is a COLLECTION of propositions

but NO universes of sets

## *in our minimalist foundation*

for minimality NO well-founded sets

$\Rightarrow$ inductive constructions

(like existence of certain inductive generated topologies)

are ADDED to the foundation when needed

## *two notions of functions from A to B*

for A, B sets

1. function as a functional relation, i.e. a proposition $R(x, y)$ s.t.

$$\forall x \in A \; \exists ! y \in B \; R(x, y)$$

2.functions as a (Bishop's) operation(= or typed theoretic function)

$$\lambda x . f(x) \in \Pi_{x \in A} B$$

operations are defined primitively!!!

## *in our minimalist foundation*

for $A, B$ sets:

FUNCTIONAL RELATIONS from A to B do NOT form a set

=Exponentiation $Fun(A, B)$ of functional relations is not a set

$$\neq$$

OPERATIONS from A to B do form a set

= Exponentiation $Op(A, B)$ is a set

*in our minimalist foundation*

for $A, B$ sets:

FUNCTIONAL RELATIONS from Nat to Nat (do NOT form a set)

are CHOICE SEQUENCES

$$\neq$$

OPERATIONS from Nat to Nat (do form a set)

are LAWLIKE SEQUENCES

## *in our minimalist foundation*

OPERATIONS from A to B do form a set

because operations ENCODE (or reflects) typed terms

$$f(x) \in B \; [x \in A]$$

which correspond to proofs of $B$ under the assumption that $x \in A$

## *why compatibility with classical predicativity*

In our minimalist foundation + excluded middle

$\boxed{\text{subsets of Nat} \quad = \quad \text{complemented subsets}}$

where

A is a complemented subset

= subset with a functional relation as characteristic map from $Nat$ to $\{0, 1\}$

A is detachable subset

= subset with an operation as characteristic map from $Nat$ to $\{0, 1\}$

but only detachable subsets form a SET!!!

## *three different kinds of reals*

at the extensional level of our minimalist foundation

| |
|---|
| reals as Dedekind cuts NOT a set |
| reals as Cauchy sequences or Bishop's reals NOT a set |
| reals as lawlike Cauchy sequences form a set |

## *why REALS do NOT form a set*

via a model

we interpret the INTENSIONAL LEVEL in CZF or ZFC as follows:

| | |
|---|---|
| sets | as subsets of natural numbers |
| | as COUNTABLE entities |
| operations between sets | as programs |
| | via Kleene realizability |
| propositions | as themselves |
| proper collections (= NO sets) | as NOT countable entities |

$$\Downarrow$$

1. Bishop or Cauchy reals or Dedekind reals are interpreted

as NOT countable entities

$\Rightarrow$ they do NOT a set

2. choice sequences of Baire and Cantor spaces do NOT form a set

# *Models of the two-level minimalist foundation*

Any model of the intensional level of our minimalist foundation
can be turned into a model of its extensional level
via a QUOTIENT COMPLETION

as in

[M.-Rosolini'12] "Quotient completion for the foundation of constructive mathematics", Logica Universalis.

[M.-Rosolini'13] ""Elementary quotient completion", Theory and applications of categories.

## *Dedekind reals as ideal points of point-free topology*

via Martin-Löf and Sambin's notion of formal topology

| |
|---|
| Dedekind reals |
| = |
| ideal points of Joyal's formal topology $\mathcal{R}_d$ |

## *point-free topology*

Martin-Löf-Sambin's formal topology

= an approach to predicative point-free topology

formal topology employs $(S, \lhd, \mathsf{Pos})$

$S$= a set of basic opens

$a \lhd U$ = a cover relation: says when a basic open $a$ is covered by the union of opens in $U \subseteq S$

FORMAL (or IDEAL) POINT= (suitable) completely prime filter

PREDICATIVE constructive POINT-FREE TOPOLOGY helps to describe the hopefully finitary (or inductive) structure of a topological space

whose points can be ONLY described in infinitary way!!

(and they do not form a set)

# pointfree presentation of Dedekind reals

Joyal's formal topology $\mathcal{R}_d \equiv (\mathbb{Q} \times \mathbb{Q}, \lhd_{\mathcal{R}}, \mathsf{Pos}_{\mathcal{R}})$

Basic opens are pairs $\langle p, q \rangle$ of rational numbers

whose cover $\lhd_{\mathcal{R}}$ is inductively generated as follows:

$$\frac{q \leq p}{\langle p, q \rangle \lhd_{\mathcal{R}} U} \qquad \frac{\langle p, q \rangle \in U}{\langle p, q \rangle \lhd_{\mathcal{R}} U} \qquad \frac{p' \leq p < q \leq q' \qquad \langle p', q' \rangle \lhd_{\mathcal{R}} U}{\langle p, q \rangle \lhd_{\mathcal{R}} U}$$

$$\frac{p \leq r < s \leq q \qquad \langle p, s \rangle \lhd_{\mathcal{R}} U \quad \langle r, q \rangle \lhd_{\mathcal{R}} U}{\langle p, q \rangle \lhd_{\mathcal{R}} U} \qquad \text{wc}\ \frac{wc(\langle p, q \rangle) \lhd_{\mathcal{R}} U}{\langle p, q \rangle \lhd_{\mathcal{R}} U}$$

where

$$wc(\langle p, q \rangle) \equiv \{\, \langle p', q' \rangle \in \mathbb{Q} \times \mathbb{Q} \mid p < p' < q' < q \}$$

## *what are Bishop reals?*

Bishop reals=regular Cauchy sequences

i.e. rational sequences $(x_n)_{x \in N^+}$ such that for $n, m \in$ N$at^+$

$$| x_n - x_m | \leq 1/n + 1/m$$

## *Bishop reals as ideal points?*

How to represent

Bishop reals= ideal points of a point-free topology

??

## Future work

- use the minimalist foundation for reverse constructive mathematics, in particular Bishop constructive analysis (w.r.t. use of Bar Induction, Fan theorem)
- relate the minimalist foundation with Feferman's systems for explicit mathematics
- study realizability models for extraction of programs from proofs