Constructive mathematics and philosophy of mathematics

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Constructive Mathematics: Foundations and practice
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Why am I interested in the philosophy of constructive mathematics?

I perceive a form of uneasiness with some aspects of traditional forms of constructivism, at least in their more common interpretations.

Some terminology:
Constructive mathematics: Bishop–style mathematics
Intuitionism/Constructivism: a family of philosophical positions which is usually associated with the use of intuitionistic logic – especially Brouwer and Dummett’s philosophy.
Two questions in the philosophy of mathematics

Do mathematical objects exist?

Is there more than one correct logic?

Intuitionism/constructivism are bound to the following philosophical positions:

- Anti–realism: there are no mind independent mathematical entities
- Logical monism: there is only one correct logic: the intuitionistic logic
Mathematical realism: the conjunction of the following

1. Existence of mathematical objects
   - platonism: abstract objects: non spatio–temporal

2. Mind independence of the mathematical objects

3. Mathematical theories are true description of the mathematical realm

Realism in ontology (1 + 2) and Realism in truth (3)

Different kinds of anti–realism follow depending on which of (1)–(3) we refute and how we do so

Anti–realists are often motivated by the difficulties realism faces on epistemology
The starting point is the philosophy of language: only a verificationist understanding of language is faithful to the way we learn and use language: \textit{meaning as use}

In the case of mathematics, truth of a statement is thus knowledge of a proof of it

A verificationist understanding of meaning and truth is incompatible with a verification–trascendent notion of truth Therefore:

Classical logic is not justified while intuitionistic logic is

We ought to be anti–realist
From the philosophy of language to intuitionistic logic and anti–realism

Dummett conflates epistemology with truth and sees an ineliminable link between the notion of truth and the ontological question

Intuitionism/Constructivism are often taken as examples of “philosophy first”: philosophy tells us how to do mathematics correctly, which methodology to use

It would seem that strictly speaking classical mathematics becomes unintelligible
Constructive mathematics vs classical mathematics: ontology

Classical logic and realism

Intuitionistic logic and anti–realism

Thus within philosophical circles the discussion between classical and constructive mathematics is often entangled with the realism vs anti–realism debate

We are faced with a strong “aut–aut”
Logical monism: there is only one correct logic

In philosophical debates logical monism is often associated with classical logic

Intuitionism/constructivism support the idea that only intuitionistic logic is correct

Again we are faced with a strict choice
Bishop’s principal motivation for developing constructive mathematics was the realisation that it enables us to uncover the computational content of classical mathematics.
In recent years Douglas Bridges and Fred Richman have proposed the following characterization of constructive mathematics: **constructive mathematics is mathematics based *only on* intuitionistic logic**

- constructive mathematics is based on intuitionistic logic
- in addition, it does not assume any further principles which are incompatible with classical mathematics

Bridges and Richman have highlighted the fact that thus constructive mathematics is a *generalization* of classical mathematics. For this reason constructive mathematics allows us to analyse mathematical concepts from a more general and refined perspective.
The idea that constructive mathematics is at the heart of a number of varieties of mathematics underlines the constructive reverse mathematics programme (Ishihara et al.)

It also underlines recent work on minimal systems for constructive mathematics (Maietti and Sambin, Aczel et al.)

Constructive mathematics is *compatibile* with classical mathematics:
- intuitionistic logic is a subsystem of classical logic
- constructive mathematics gives us a better perspective
We need to access classical mathematics, as we wish to analyse it and compare it with other kinds of mathematics.

Classical mathematics is not only fully intelligible but it is also a source of inspiration for the constructive mathematician.

Therefore the constructive mathematical practice seems to suggest a different, more conciliatory view of the relation between classical and constructive mathematics.
We start from the mathematical practice rather than the philosophy of mathematics and ask different questions:

Why do we do mathematics constructively?

Because:

- it is computational
- it is more general
- and thus allows us to be faithful to distinctions between classically equivalent notions
- its proofs are more informative
- its proofs are more explicit
- its proofs are sometimes simpler, even shorter, easier to understand??
- ...

Laura Crosilla  University of Leeds  Constructive set theory
Mathematical reasons for constructive mathematics

That is: there are mathematical (and epistemological?) reasons for doing mathematics constructively

Such reasons seem to be bound to the methodology of constructive mathematics, which is centered around the notion of constructive proof

What is the benefit of this change of perspective?

It would seem that we are now not bound to specific philosophical positions either on ontology or on philosophy of logic

In this way we can start addressing more interesting questions like: what is (constructive) mathematics, what are its methodology, its heuristics, etc.?
Already Heyting expressed the mathematician’s dream that his practice would not require him to take specific metaphysical positions on the nature of the mathematical objects:

“We have no objections against a mathematician privately admitting any metaphysical theory he likes, but Brouwer’s program entails that we study mathematics as something simpler, more immediate than metaphysics” Heyting 1956, p. 2
Ontological neutrality of constructive mathematics?

Does constructive mathematics force us on the anti–realist camp?

One way to claim that it does not, would be to show that it is compatible with the first two component of realism (realism in ontology)

This route is now open to us because anti–realism is not embedded into the whole picture from the start

A structuralist view seems quite fitting here
A mathematical structure is investigated using different tools: constructive and intuitionistic logic.

The use of different logics corresponds to different structures, where the definitions of the structures include indication of which logic is to be used to study them.

Fred Richman has proposed the first metaphor: one kind of mathematical objects studied from a number of different perspectives (constructive, intuitionistic, classical etc), with the constructive one offering the most general view.

The second metaphor is in agreement with Stewart Shapiro’s version of mathematical structuralism.
Fred Richman and David McCarty have claimed that there is no difference in language between classical and intuitionistic logic, contrary to very well known philosophical interpretations.
Logical monism is a view for which there is only one correct logic

Logical contextualism claims that different logics can be seen as correct in different contexts
This is a view often associated with Carnap’s principle of tolerance (and Quine)

This positions is bound with the idea that classical and intuitionistic logic are embedded in *different languages*: there are classical and constructive connectives and quantifiers
For Carnap, the choice of the language is an external question, which is settled on the basis of pragmatic reasons
In recent years Beall and Restall have promoted a view of logic called Logical pluralism.

It focuses on the notion of logical consequence and its main claim is that there are different *equally good* ways of specifying such notion: a classical and an intuitionistic ones among them.

Thus classical and intuitionistic logic are both equally justified.
Logical pluralism or logical contextualism?

Logical pluralism sees the differences between classical and intuitionistic logic as arising from different renderings of the notion of logical consequence, not from differences in the meaning of the logical constants.

In simple terms: logical pluralism sees classical derivations as of the form:

\[ \varphi \vdash_c \psi \]

and intuitionistic derivations as of the form:

\[ \varphi \vdash_i \psi \]

Logical relativism as: \( \varphi_c \vdash \psi_c \) and \( \varphi_i \vdash \psi_i \), respectively.

Kuhn on incommensurability.
Why constructive mathematics?

Logical pluralism suggests that a number of logics are equally good ways of rendering the notion of consequence relation.

Logical contextualism suggests that pragmatic considerations suggest which of a number of languages is preferable over the others.

Even if Richman has proposed ideas which well fit with pluralism, he does not claim that classical and constructive mathematics are equally good.

The constructive mathematician seems to think that there is a sense in which constructive mathematics is *better* than classical mathematics.
A traditional philosophical route to intuitionistic logic seems to force us to accept anti–realism. It also seems to force us to accept only the intuitionistic logic as correct.

If we wish to grant more to classical mathematics, an analysis of the mathematical practice might offer a more promising start: it is an attempt to address questions on the nature of (constructive) mathematics by looking at its methodology, its motivation, its heuristics etc.

It also seems to leave open the route to different answers to the realism vs anti–realism debate and to the question on the nature of logic. From there we can access a wider choice of options, if we wish to engage with “external” questions.
Learning from the philosophy of science

Context of discovery and context of justification
Some papers I have found inspiring