Unified Approach to Real Numbers in Various Mathematical Settings

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Overview

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- Classically equivalent definitions/constructions of **reals** $\mathbb{R}$ differ constructively. The choice of reals is made depending on the setting — no unifying definition.
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Purpose of the talk:

- provide a setting-independent definition of reals $\mathbb{R}$ by introducing **streaks**,
- show that standard constructions of reals satisfy our definition in their respective settings,
- study the structure of reals by noting that its pieces correspond to reflections on the category of streaks,
- observe that our definition enables us to do constructive analysis without refering to a specific model of reals.

Contents

- **Streaks**
  We define streaks and with them characterize number sets, including $\mathbb{R}$.
- **Reflections**
  We observe that pieces of structure of reals correspond to reflections on the category of streaks.
- **Models**
  We observe that the standard constructions of reals satisfy our definition. Moreover, our theory provides explicit formulae for pieces of structure in each particular construction.

To do this, we use the language of *category theory*, particularly the *universal property*.

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- As an additional degree of freedom we assume that sets have **intrinsic topology**:
  - For every set X we have the classes of "open" and "closed" subsets $\mathcal{O}(X), \mathcal{Z}(X) \subseteq \mathcal{P}(X)$.
  - All maps between sets are continuous with regard to this topology.
  - Open sets are closed under countable unions and finite intersections.
  - Closed sets are closed under countable intersections and doubly complemented finite unions.
  - Disjoint unions of sets and quotients have the expected topology.

It follows that every decidable subset is open and closed. Hence classically the only possible intrinsic topology is the discrete one.

Streaks

Informal definition: A **streak** is a

- strict "linear" archimedean order
- with as much algebraic structure as preserves this order (addition, as well as multiplication of positive elements)
- such that $<$ is open and $\leq$ closed.

Intuition: Being a linear order forces a streak to lie on a line; being additionally archimedean forces it to lie on its finite part, i.e. on the real line. Hence $\mathbb{R}$ can be characterized as the largest (in categorical terms, terminal) streak.

Definition: $(X, <, +, 0, \cdot, 1)$ is a **streak** when

- $<$ is an asymmetric and cotransitive binary relation on X
  (hence $a \# b := a < b \vee b < a$ is an apartness and $a \leq b := \neg(b < a)$ a preorder),
- $\#$ is tight (equivalently, $\leq$ is a partial order),
- $(X, +, 0)$ is a commutative monoid
  (therefore we can multiply elements of X with natural numbers),
- $(X_{>0}, \cdot, 1)$ is a commutative monoid and $\cdot$ distributes over $+$,
- $a + x < b + x \iff a < b$ for all $a, b, x \in X$,
- $a \cdot x < b \cdot x \iff a < b$ for all $a, b, x \in X_{>0}$,
- $<$ is an open and $\leq$ a closed subset of $X \times X$,
- the **archimedean condition** holds: for all $a, b, c, d \in X$ with $b < d$ there exists $n \in \mathbb{N}$ with $a + n \cdot b < c + n \cdot d$.

Definition: A map $f \colon X \to Y$ is a **streak morphism** from $(X, <, +, 0, \cdot, 1)$ to $(Y, <, +, 0, \cdot, 1)$ when it preserves all the structure, i.e.

- $a < b \implies f(a) < f(b)$,
- $f(a + b) = f(a) + f(b), \quad f(0) = 0$,
- $f(a \cdot b) = f(a) \cdot f(b), \quad f(1) = 1$.

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$\mathbb{N}$ is the initial ("smallest") streak, $\mathbb{Z}$ is the initial ring streak, $\mathbb{Q}$ is the initial field streak.

Definition: $\mathbb{R}$ is the terminal streak (i.e. for every streak X there exists a (unique) streak morphism $!_X : X \to \mathbb{R}$).

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Lemma: If $\underline{\mathbf{R}} \subseteq \underline{\mathbf{C}}$ is a reflective subcategory and $\underline{\mathbf{C}}$ has a terminal object 1, then $R(1) \cong 1$ and $R(1)$ is a terminal object in both $\underline{\mathbf{C}}$ and $\underline{\mathbf{R}}$.

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Corollary: $\mathbb{R}$ has every reflective structure.

Examples of reflections

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Theorem: For any streak X the sets $X^\wedge$ and $X^\vee$ of inhabited finite subsets of X, quotiented by a suitable equivalence relation, are streaks, closed under binary infima and suprema respectively. They define reflections from streaks to semilattice streaks which commute with each other up to isomorphism. Their composition gives a reflection from streaks to lattice streaks.

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$$|a| := \sup\{a, -a\}$$

and hence the **euclidean metric**

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As usual the **metric balls** are given by

$$B(a, r) := \{x \in \mathbb{R} \mid d(a, x) < r\}.$$

Since $<$ is open, so are the balls and the intrinsic topology of $\mathbb{R}$ is always at least as strong as the euclidean topology.

Models of reals

Concrete models of reals are typically given as some sort of completion of rationals. However, other dense sets (such as dyadic rationals) work as well.

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In any streak X define for $q \in \mathbb{Q}$, $x \in X$

$$q < x := i < j + k \cdot x$$

where $q = \frac{i-j}{k}$, $i, j \in \mathbb{N}$, $k \in \mathbb{N}_{>0}$. Similarly for $x < q$.

Definition: A streak X is **dense** when for every $q, r \in \mathbb{Q}$ with $q < r$ there exists $x \in X$ with $q < x < r$.

Also, for any two streak X, Y define $x < y := \exists\, q \in \mathbb{Q} \,.\, x < q < y$ for $x \in X$, $y \in Y$.

Cauchy reals

Lemma: For any streak X the set of its *Cauchy sequences* $CS(X)$ is a "non-tight streak" and the set of their equivalence classes $CC(X)$ (the "Cauchy completion" of X) is a streak. The embedding of X as constant sequences is a streak morphism.

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Definition: A streak X is called **Cauchy complete** when this embedding $\gamma_X : X \to CC(X)$ is an isomorphism.

Theorem: If countable choice holds, then CC is a reflection of streaks into Cauchy complete streaks, and for any dense streak X the streak $CC(X)$ is terminal — thus a model of $\mathbb{R}$.

Idea of proof: $f : Y \to CC(X)$ is defined $f(y) := [a]$ where a_n is chosen in X in the way that $2n \cdot a_n < 2n \cdot y + 1$ and $2n \cdot y < 2n \cdot a_n + 1$.

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Consider the following diagram of "not necessarily tight streaks" and their morphisms.

$$\begin{tikzcd}
 & CS(\mathbb{R}) \arrow[d, "\theta_{CS(\mathbb{R})}"] \\
\mathbb{R} \arrow[ur, "c_{\mathbb{R}}"] \arrow[r, "\cong", "\gamma_{\mathbb{R}}"'] & CC(\mathbb{R})
\end{tikzcd}$$

Define $\lim_{\mathbb{R}} := \gamma_{\mathbb{R}}^{-1} \circ \theta_{CS(\mathbb{R})} = !_{CC(\mathbb{R})} \circ \theta_{CS(\mathbb{R})}$.

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Theorem: We have $\lim_{\mathbb{R}} \circ c_{\mathbb{R}} = \mathrm{Id}_{\mathbb{R}}$, $\lim_{\mathbb{R}}(a+b) = \lim_{\mathbb{R}}(a) + \lim_{\mathbb{R}}(b)$, $\lim_{\mathbb{R}}(a \cdot b) = \lim_{\mathbb{R}}(a) \cdot \lim_{\mathbb{R}}(b)$, and $\lim_{\mathbb{R}}$ satisfies the usual definition of a limit:

$$x = \lim_{\mathbb{R}}(a) \iff \forall \epsilon \in \mathbb{R}_{>0} \,.\, \exists n \in \mathbb{N} \,.\, \forall i \in \mathbb{N}_{\geq n} \,.\, d(x, a_i) < \epsilon.$$

Dedekind reals

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Theorem: If $\mathcal{O}(X)$ is a set, then so is $\mathcal{D}(X)$. If X is furthermore a dense streak, then $\mathcal{D}(X)$ is a terminal streak — thus a model of $\mathbb{R}$.

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Remark: A related construction is to give reals via the **interval domain**. Taking open intervals is just a rephrasement of Dedeking reals, so it works as above. Taking closed intervals works too, except for the topological conditions; it works if we e.g. postulate discrete intrinsic topology overall.

Reals as a formal space/locale/classical topological space

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Theorem:

- The formal space of reals is a terminal streak in the category of formal spaces.
- The locale of reals is a terminal streak in the category of locales.
- The topological space of reals is a terminal streak in the category of topological spaces.

Idea of proof (for topological spaces): View $\mathbb{R}$ as a formal space, its basis $\overline{\mathbb{Q}} \times \overline{\mathbb{Q}}$ given by "(possibly infinite) rational intervals". Define $f : Y \to \mathbb{R}$ by $f(y) := \big\{(q, r) \in \overline{\mathbb{Q}} \times \overline{\mathbb{Q}} \;\big|\; q < y < r\big\}$.

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Note: <u>terminality implies that $\mathbb{R}$ must have the euclidean topology!</u>

Concluding remarks

- We have seen that the definition of $\mathbb{R}$ as the terminal streak works independently of the setting.
- We have seen that reflections on the category of streaks (including the field and the lattice structure) not only equip $\mathbb{R}$ with additional structure on the theoretical level, but also provide explicit formulae for this structure in specific models of $\mathbb{R}$.
- In particular, the existence of the absolute value, the euclidean metric and the limit operator on $\mathbb{R}$ follows from the definition. We can do constructive analysis without using a specific model of $\mathbb{R}$.
- Similar ideas can be used to characterize lower reals, upper reals or metric completions.

Other authors have also used universal property to define reals/intervals:

- **D. Pavlović** and **V. Pratt** (Paper: On coalgebra of real numbers) give the interval $\mathbb{R}_{[0,1)}$ as the terminal coalgebra of the functor $X \mapsto X \cdot \omega$.
  - Doesn't work constructively.
  - Gives a semiclosed interval rather than $\mathbb{R}$ directly.
- **P. Freyd** (Paper: Algebraic real analysis) uses the midpoint operation to give the interval $\mathbb{R}_{[0,1]}$ as the terminal coalgebra for $X \mapsto X \vee X$.
  - The original version uses classical logic, but Freyd checks that a modified definition is satisfied by reals via signed digit representation assuming dep. choice, and by Dedekind reals in sheaf topoi. Does it work in general?
  - Can other operations be defined?
  - Gives a closed interval rather than $\mathbb{R}$ directly.
- **M. Escardó** and **A. Simpson** (Paper: A universal characterization of the closed euclidean interval) use the infinitary version of the midpoint operation to give the interval $\mathbb{R}_{[0,1]}$ as the free cancellative iterative midpoint object over two generators.
  - Works constructively, assuming countable choice (it gives the Cauchy completion of diadic rationals on a closed interval).
  - Gives a closed interval rather than $\mathbb{R}$ directly.

My thanks to the organizers

for the invitation.