Intuitionistic Probability Logics

Zoran Marković

Matematički institut SANU
zoranm@mi.sanu.ac.rs

Niš, 24. 6. 2013.

Background

- Leibnitz:
  Quod facile est in re, id probabile est in mente (that which is easy in the thing, that is probable in the mind)
  Our judgment of probability (in the mind) is proportional to what we believe (to be propensity in things). Probability is here degree of certainty.
- Jacob Bernoulli (Ars conjectandi, 1713):
  Probability is degree of certainty.
  Even if events A and B are disjoint, the probability $P(A \cup B)$ need not be the sum of the probabilities of A and B.

The development of probabilistic logic through history

- Through history, logic was paying equal attention to both types of truth - necessary and contingent - until the beginning of the XX century.
- Here are some of the famous mathematicians who investigated the problem of probabilistic inference:

Gotfried W. Leibnitz (1646 - 1716) Jacobus Bernoulli (1654 - 1705)
Johann Bernoulli (1667 - 1748) Thomas Bayes (1702 - 1761)
Johann Heinrich Lambert (1728 - 1777) Bernard Bolzano (1781 - 1848)
Pierre Simon de Laplace (1749 - 1827) Augustus De Morgan (1806 - 1871)
George Boole (1815 - 1864) John Venn (1834 - 1923)
Hugh MacColl (1837 - 1909) Charles S. Pierce (1839 - 1887)
Hans Reichenbach (1891 - 1953) John M. Keynes (1883 - 1946)
Rudolph Carnap (1891 - 1970)

George Boole

p - a proposition $\Longleftrightarrow$ set of all cases (possible worlds) when p is true

Probability as a semantics

- Mostowsky, Rasiowa, Sikorski: boolean and pseudo-boolean valued models (early '50's)
- Gaifman, Scott, Krauss: probability as truth-values ('60's)
- Morgan, Leblanc: same for intuitionism (1983)
  define a notion of probability such that $B \vdash A$ iff $Pr(A, B) = 1$
- Roeper, Leblanc: same for $\vdash A$ and $Pr(A) = 1$ (1999)
  since there are less int. theorems than classical, there must be more int.prob. functions
- Weatherson (2003): proposes intuitionistic Bayesianism as an answer to the criticism of the classical Bayesianism (e.g., additivity of probability),
  gives philosophical objections to Morgan and Leblanc axioms, e.g., $Pr(A, B \wedge C) \geq Pr(A, B)$ ("no negative evidence" rule)

Probability as a part of syntax (1)

- Hans Reichenbach: $A \rightarrow_p B$ (1930's)
- Jerome Keisler: probability quantifiers (1970's)
- R. Fagin, J. Halpern and N. Megiddo (1990): probabilistic weight functions as a part of syntax, with Kripke semantics, for a weakly complete system
- Rašković and Boričić (1996): probability as propositional (modal-like) operators added to intuitionistic propositional logic
  semantics: Kripke-model where each world has a pair of probability measures (lower and upper) which converge as we climb up the tree, weak completeness

Probability as a part of syntax (2)

Zoran Marković, Zoran Ognjanović, Miodrag Rašković, An intuitionistic logic with probabilistic operators, Publications de L'Institute Matematique, n.s. 73 (87), 31 - 38, 2003.

Zoran Marković, Zoran Ognjanović, Miodrag Rašković, A Probabilistic Extension of Intuitionistic Logic, Mathematical Logic Quarterly, vol. 49, 415-424, 2003.

- much simpler system - more in line with Boole's ideas,
- no iterations of probabilistic operators
- first proof of strong completeness - using infinitary rules inherent non-compactness:

$$F = \{\neg P_{=0}\alpha\} \cup \{P_{<1/n}\alpha : n \text{ is a positive integer}\}$$

Language

- S - the unit interval of rational numbers
- $\mathrm{Var} = \{p, q, r, \ldots\}$ (propositional letters), connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$
- For_I is the set of intuitionistic propositional formulas
- Basic probabilistic formulas:
  - $P_{\geq s}\alpha$ for $\alpha \in For_I$, $s \in S$,
  - $P_{\leq s}\alpha$ for $\alpha \in For_I$, $s \in S$.
- For_P = Boolean combinations of basic probabilistic formulas

Axioms

(1) all For_I-instances of intuitionistic propositional tautologies

(2) all For_P-instances of classical propositional tautologies

(P1) $P_{\geq 0}\alpha$

(P2) $P_{\geq 1-r}\neg\alpha \to \neg P_{\geq s}\alpha$, for $s > r$

(P3) $P_{\geq r}\alpha \to P_{\geq s}\alpha$, for $r \geq s$

(P4) $P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta)$

(P5) $(P_{\geq s}\alpha \wedge P_{\geq r}\beta \wedge P_{\geq 1}\neg(\alpha \wedge \beta)) \to P_{\geq \min(1,s+r)}(\alpha \vee \beta)$

(P6) $(P_{<s}\alpha \wedge P_{<r}\beta) \to P_{<s+r}(\alpha \vee \beta)$, $s + r \leq 1$

(P7) $P_{\leq 1}\alpha$

(P8) $P_{\geq r}\alpha \to \neg P_{\leq s}\alpha$, $s < r$

(P9) $\neg P_{\geq r}\alpha \to P_{\leq s}\alpha$, $r < s$

Inference Rules

(R1) From φ and $\varphi \to \psi$ infer ψ.

(R2) If $\alpha \in For_I$, from α infer $P_{\geq 1}\alpha$.

(R3) From $B \to P_{\geq s-\frac{1}{k}}\alpha$, for every integer $k \geq \frac{1}{s}$, infer $B \to P_{\geq s}\alpha$.

(R4) From $B \to P_{\leq s+\frac{1}{k}}\alpha$, for every integer $k \geq \frac{1}{1-s}$, infer $B \to P_{\leq s}\alpha$.

Semantics

An intuitionistic (propositional) Kripke model for the language For_I is a structure $\langle W, \leq, v \rangle$ where:

- $\langle W, \leq \rangle$ is a partially ordered set of possible worlds which is a tree, and
- v is a valuation function, i.e., v maps the set W into the powerset $\mathcal{P}(\mathrm{Var})$, which satisfies the condition: for all w, $w' \in W$, $w \leq w'$ implies $v(w) \subseteq v(w')$.

Note: in any classical possible-world model we may introduce intuitionistic connectives by first defining the ordering of the worlds by:

$$w \leq w' \text{ iff } v(w) \subseteq v(w')$$

The forcing relation $\Vdash$

Let $\langle W, \leq, v \rangle$ be an intuitionistic Kripke model. The forcing relation $\Vdash$ is defined by the following conditions for every $w \in W$, $\alpha, \beta \in For_I$:

- if $\alpha \in \mathrm{Var}$, $w \Vdash \alpha$ iff $\alpha \in v(w)$,
- $w \Vdash \alpha \wedge \beta$ iff $w \Vdash \alpha$ and $w \Vdash \beta$,
- $w \Vdash \alpha \vee \beta$ iff $w \Vdash \alpha$ or $w \Vdash \beta$,
- $w \Vdash \alpha \rightarrow \beta$ iff for every $w' \in W$ if $w \leq w'$ then $w' \nVdash \alpha$ or $w' \Vdash \beta$, and
- $w \Vdash \neg\alpha$ iff for every $w' \in W$ if $w \leq w'$ then $w' \nVdash \alpha$.

Validity

- Validity in the intuitionistic Kripke model $\langle W, \leq, v \rangle$ is defined by

$$\langle W, \leq, v \rangle \models \alpha \text{ iff } (\forall w \in W) w \Vdash \alpha.$$

- A formula α is valid ($\models \alpha$) if it is valid in every intuitionistic Kripke model.

Theorem. For every intuitionistic Kripke model $\langle W, \leq, v \rangle$, every $w \in W$, and every $\alpha \in \mathit{For}_I$ the following holds: $w \Vdash \alpha$ iff for every $w' \in W$ if $w \leq w'$ then $w' \Vdash \alpha$.

Theorem. If 0 is the least element in $\langle W, \leq \rangle$, then

$$\langle W, \leq, v \rangle \models \alpha \quad \text{iff} \quad 0 \Vdash \alpha.$$

Additional notation

- $M = \langle W, \leq, v \rangle$ – an intuitionistic Kripke model
- for every $\alpha \in For_I$, $[\alpha]_M = \{w \in W : w \Vdash \alpha\}$
- $H_I = \{[\alpha]_M : \alpha \in For_I\}$ is a Heyting algebra with operations
  - $[\alpha]_M \cup [\beta]_M = [\alpha \vee \beta]_M$,
  - $[\alpha]_M \cap [\beta]_M = [\alpha \wedge \beta]_M$,
  - $[\alpha]_M \Rightarrow [\beta]_M = [\alpha \rightarrow \beta]_M$,
  - $\sim [\alpha]_M = [\neg\alpha]_M$
- H_I is a lattice on W, but it may not be closed under complementation

Interpretation of probabilistic operators (1)

A probabilistic model is a structure $\langle W, \leq, v, H, \mu \rangle$ where:

- $\langle W, \leq, v \rangle$ is an intuitionistic Kripke model,
- H is an algebra on W containing $H_I = \{[\alpha]_M : \alpha \in For_I\}$,
- $\mu : H \to [0, 1]$ is a finitely additive probability.

Note:

- it is possible that $W \setminus [\alpha]_M \neq [\neg\alpha]_M$
- both $P_{\geq s}$ and $P_{\leq s}$ are needed since $P_{\leq s}\alpha$ does not imply $P_{\geq 1-s}\neg\alpha$.
- IP denotes the class of all probabilistic models

Interpretation of probabilistic operators (2)

The satisfiability relation $\models$ is defined by the following conditions for every probabilistic model $M = \langle W, \leq, v, H, \mu \rangle$:

- if $\alpha \in For_I$, $M \models \alpha$ if $(\forall w \in W) w \Vdash \alpha$,
- $M \models P_{\geq s}\alpha$ if $\mu([\alpha]_M) \geq s$,
- $M \models P_{\leq s}\alpha$ if $\mu([\alpha]_M) \leq s$,
- if $A \in For_P$, $M \models \neg A$ if $M \models A$ does not hold, and
- if $A, B \in For_P$, $M \models A \wedge B$ if $M \models A$, and $M \models B$.

Formula satisfiability and validity

- $\varphi \in For$ is satisfiable if there is a probabilistic model M such that $M \models \varphi$
- φ is valid if for every probabilistic model M, $M \models \varphi$
- a set of formulas is satisfiable if there is a probabilistic model M such that for every formula φ from the set, $M \models \varphi$

Results

Theorem (Soundness theorem)

the axiomatic system Ax_{IP} is sound with respect to the class IP.

Theorem (Extended completeness theorem)

Every Ax_{IP}-consistent set of formulas is IP-satisfiable.

Theorem (Decidability theorem)

The IP-satisfiability problem is decidable.

Examples (1)

It is well known that

$$\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$$

is a classical tautology, called De Morgan's law, which is not an intuitionistic tautology. Still, even if we believe that it is impossible to have your cake (p) and eat it (q), we do not believe that it is impossible to have your cake and we also do not believe that it is impossible to eat your cake. More formally, we would like to have

$$P_{\geq 1}\neg(p \wedge q), \qquad P_{\leq \epsilon}\neg p, \qquad P_{\leq \epsilon}\neg q$$

for come small ϵ, which is impossible with the classical logic.

Examples (2)

Consider the classical (but not intuitionistic) tautology

$$(p \to q) \vee (q \to p)$$

Starting with classical logic makes

$$\models P_{\geq 1}((p \to q) \vee (q \to p)).$$

Let $p =$ it rains, $q =$ the sprinkler is on
The sprinkler should not be on when it rains so $p \to q$ should have low probability, say less than ϵ, so:

$$P_{\leq \epsilon}(p \to q)$$

Since probability is additive, the probability of $q \to p$ has to be high, i.e., we get that it is very probable that it will rain whenever the sprinkler is on

$$P_{\geq 1-\epsilon}(q \to p)$$

which certainly would not be a desirable consequence.

Possible extensions

- Logic with constructive probability: $\nvdash P_{\geq s}\alpha \vee \neg P_{\geq s}\alpha$
- Logics with richer languages:
  - conditional probability operators $CP_{\geq s}(A, B)$
  - a operator for qualitative probability $A \lhd B$
  - first order probability logics
  - iterations of probabilistic operators
- Different ranges of probabilistic functions:
  - finite ranges $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$
  - infinitesimals: $CP_{\approx 1}(A, B)$
  - ...

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