Constructive reverse mathematics: an introduction

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CMFP 2013, Nis, 24-28 June, 2013
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A history of constructivism

- History
  - Arithmetization of mathematics (Kronecker, 1987)
  - Three kinds of intuition (Poincaré, 1905)
  - French semi-intuitionism (Borel, 1914)
  - Intuitionism (Brouwer, 1914)
  - Predicativity (Weyl, 1918)
  - Finitism (Skolem, 1923; Hilbert-Bernays, 1934)
  - Constructive recursive mathematics (Markov, 1954)
  - Constructive mathematics (Bishop, 1967)

- Logic
  - Intuitionistic logic (Heyting, 1934; Kolmogorov, 1932)
Mathematical theory

A mathematical theory consists of
- axioms describing mathematical objects in the theory
- logic being used to derive theorems from the axioms

Two ways of generalizing mathematical theory:

<table>
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We use the standard language of (many-sorted) first-order predicate logic based on
- primitive logical operators $\land, \lor, \rightarrow, \bot, \forall, \exists$.

We introduce the abbreviations
- $\neg A \equiv A \rightarrow \bot$;
- $A \leftrightarrow B \equiv (A \rightarrow B) \land (B \rightarrow A)$.
Deduction

We shall use $\mathcal{D}$, possibly with a subscript, for arbitrary deduction. We write

\[
\Gamma \vdash \mathcal{D} \quad A
\]

to indicate that $\mathcal{D}$ is deduction with conclusion $A$ and assumptions $\Gamma$. 

Deduction (Basis)

For each formula $A$, $A$ is a deduction with conclusion $A$ and assumptions $\{A\}$. 
Deduction (Induction step)

If

\[ \Gamma \vdash B \]

is a deduction, then

\[
\frac{\Gamma}{D} \\
B \\
\frac{B}{A \rightarrow B} \rightarrow \text{I}
\]

is a deduction with conclusion \( A \rightarrow B \) and assumptions \( \Gamma \setminus \{A\} \).

We write

\[
\frac{[A]}{D} \\
B \\
\frac{B}{A \rightarrow B} \rightarrow \text{I}
\]
Deduction (Induction step)

If

\[
\begin{array}{c}
\Gamma_1 \\
D_1 \\
A \rightarrow B \\
\end{array}
\quad
\begin{array}{c}
\Gamma_2 \\
D_2 \\
A \\
\end{array}
\]

are deductions, then

\[
\begin{array}{c}
\Gamma_1 \\
D_1 \\
A \rightarrow B \\
\end{array}
\quad
\begin{array}{c}
\Gamma_2 \\
D_2 \\
A \\
\end{array}
\]

\[
\frac{A \rightarrow B \quad A}{B \quad \rightarrow E}
\]

is a deduction with conclusion $B$ and assumptions $\Gamma_1 \cup \Gamma_2$. 
Deduction (Induction step)

If

\[
\begin{array}{ccc}
\Gamma_1 & \Gamma_2 & \Gamma_3 \\
\mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \\
A \lor B & C & C
\end{array}
\]

are deductions, then

\[
\begin{array}{ccc}
\Gamma_1 & \Gamma_2 & \Gamma_3 \\
\mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \\
A \lor B & C & C
\end{array}
\]

\[
\frac{A \lor B}{C} \quad \lor E
\]

is a deduction with conclusion \( C \) and assumptions \( \Gamma_1 \cup (\Gamma_2 \setminus \{A\}) \cup (\Gamma_3 \setminus \{B\}) \).

We write

\[
\begin{array}{ccc}
\mathcal{D}_1 & [A] & [B] \\
\mathcal{D}_2 & \mathcal{D}_3 \\
A \lor B & C & C
\end{array}
\]

\[
\frac{A \lor B}{C} \quad \lor E
\]
Minimal logic

\[
\begin{align*}
\text{\(D_1\)} & \quad \text{\(D_2\)} \\
\frac{A}{A \land B} & \quad \frac{B}{A \land B} \\
\hline
& \land I \\
\text{\(D\)} & \\
\frac{A}{A \lor B} & \quad \frac{B}{A \lor B} \\
\hline
& \lor I_r \\
[D] & \\
\frac{D}{A} & \quad \frac{B}{A} \\
\hline
& \rightarrow I \\
\text{\(D\)} & \\
\frac{A \land B}{A} & \quad \frac{A \land B}{B} \\
\hline
& \land E_r \\
\text{\(D\)} & \\
\frac{A \land B}{A} & \quad \frac{A \land B}{B} \\
\hline
& \land E_I \\
[A] & \quad [B] \\
\frac{D_1}{A \lor B} & \quad \frac{D_2}{C} & \quad \frac{D_3}{C} \\
\hline
& \lor E \\
\text{\(D\)} & \\
\frac{D_1}{A \rightarrow B} & \quad \frac{D_2}{A} \\
\hline
& \rightarrow E
\end{align*}
\]
Minimal logic

\[
\frac{D}{A} \quad \forall I \\
\frac{\forall yA[x/y]}{A[\langle x \rangle]} \quad \forall I
\]

\[
\frac{D}{A[\langle x \rangle]} \quad \forall E
\]

\[
\frac{D}{A[\langle x \rangle]} \quad \exists I
\]

\[
\frac{\exists yA[x/y]}{C} \quad \exists E
\]

- In \( \forall E \) and \( \exists I \), \( t \) must be free for \( x \) in \( A \).
- In \( \forall I \), \( D \) must not contain assumptions containing \( x \) free, and \( y \equiv x \) or \( y \not\in \text{FV}(A) \).
- In \( \exists E \), \( D_2 \) must not contain assumptions containing \( x \) free except \( A \), \( x \not\in \text{FV}(C) \), and \( y \equiv x \) or \( y \not\in \text{FV}(A) \).
Minimal logic

We denote by

$$\Gamma \vdash_m A$$

that there is a deduction in minimal logic with the conclusion $A$ and the assumptions in $\Gamma$. 
Example (minimal logic)

\[
\frac{\neg B}{B} \quad \frac{\neg A}{E}
\]

\[
\frac{\neg \neg A}{\neg A} \quad \frac{\bot}{\neg A} \quad \frac{\bot}{E}
\]

\[
\frac{\neg \neg (A \rightarrow B)}{\neg (A \rightarrow B)} \quad \frac{\bot}{\neg \neg B} \quad \frac{\bot}{\neg \neg A \rightarrow \neg \neg B} \quad \frac{\bot}{(\neg \neg A \rightarrow \neg \neg B)}
\]
Intuitionistic logic

Intuitionistic logic is obtained from minimal logic by adding the intuitionistic absurdity rule:

\[
\begin{array}{c}
\exists \ \bot \\
\hline \\
A \\
\bot_i
\end{array}
\]

We denote by

\[\Gamma \vdash_i A\]

that there is a deduction in intuitionistic logic with the conclusion \(A\) and the assumptions in \(\Gamma\).

Note that

\[\Gamma \vdash_m A \Rightarrow \Gamma \vdash_i A.\]
Example (intuitionistic logic)

\[
\begin{align*}
\neg
\neg A & \rightarrow \neg
\neg B \\
\neg
\neg (A \rightarrow B) & \rightarrow \neg
\neg (A \rightarrow B)
\end{align*}
\]
Classical logic

Classical logic is obtained from intuitionistic logic by strengthening the absurdity rule to the classical absurdity rule:

\[
\begin{array}{c}
\neg A \\
\Downarrow \\
A \quad \bot_c
\end{array}
\]

We denote by

\[\Gamma \vdash_c A\]

that there is a deduction in classical logic with the conclusion \(A\) and the assumptions in \(\Gamma\).

Note that

\[\Gamma \vdash_i A \Rightarrow \Gamma \vdash_c A.\]
The double negation elimination (DNE):

\[
\frac{\neg \neg A}{\neg A} \quad \frac{\neg A \quad \perp_c}{A} \quad \rightarrow \text{E}
\]

\[
\frac{\perp}{A} \quad \frac{A}{\neg \neg A \rightarrow A} \quad \rightarrow \text{I}
\]
The principle of excluded middle (PEM):

\[
\begin{align*}
\neg(A \lor \neg A) & \quad \frac{[A]}{A \lor \neg A} \quad \lor \text{I}_r \\
\frac{\bot}{A \lor \neg A} & \quad \rightarrow \text{I} \\
\frac{\neg A}{A \lor \neg A} & \quad \lor \text{I}_l \\
\frac{\bot}{A \lor \neg A} & \quad \rightarrow \text{E} \\
\end{align*}
\]
Example (classical logic)

De Morgan’s law (DML):

\[
\neg (A \land B) \rightarrow \neg A \lor \neg B
\]
Logical equivalences as equivalence relations

Each logic defines an equivalence relation $\sim_*$ on the set of formulas by

$$A \sim_* B \iff \vdash_* A \leftrightarrow B$$

where $* \in \{m, i, c\}$. We have

$$A \sim_m B \Rightarrow A \sim_i B \Rightarrow A \sim_c B$$

and

- $\neg \neg \neg A \in [\neg A]_m$
- $\neg \neg A \rightarrow \neg \neg B \not\in [\neg \neg (A \rightarrow B)]_m$
- $\neg \neg A \rightarrow \neg \neg B \in [\neg \neg (A \rightarrow B)]_i$
- $\neg (A \land B) \not\in [\neg A \lor \neg B]_i$
- $\neg (A \land B) \in [\neg A \lor \neg B]_c$
Notations

- $m, n, i, j, k, \ldots \in \mathbb{N}$
- $a, b, c, \ldots \in \mathbb{N}^*$
  - $|a|$ = the length of $a$
  - $a \ast b$ = the concatenation of $a$ and $b$
  - $a \preceq b \iff \exists c (a \ast c = b)$
- $\alpha, \beta, \gamma, \delta, \ldots \in \mathbb{N}^\mathbb{N}$
  - $\overline{\alpha}(n) = (\alpha(0), \ldots, \alpha(n-1))$
  - $\alpha \in a \iff \overline{\alpha}(|a|) = a$
- $\{0, 1\}^* = \{a \mid \forall n < |a| (a(n) = 0 \lor a(n) = 1)\}$
- $\{0, 1\}^\mathbb{N} = \{\alpha \mid \forall n (\alpha(n) = 0 \lor \alpha(n) = 1)\}$
Axioms of intuitionism

- The weak continuity for numbers (\textit{WC-N}):

\[ \forall \alpha \exists n A(\alpha, n) \rightarrow \forall \alpha \exists mn \forall \beta \in \alpha(m) A(\beta, n) \]

- The fan theorem (\textit{FAN})�

\[ \forall \alpha \in \{0, 1\}^N \exists n A(\alpha(n)) \rightarrow \exists n \forall \alpha \in \{0, 1\}^N \exists k \leq n A(\alpha(k)) \]

We denote \textit{FAN} for quantifier free \( A \) by \textit{FAN}_D.
Axioms of intuitionism (continuity principle)

Let $F$ be a function from $\mathbb{N}^\mathbb{N}$ into $\mathbb{N}$. Then

- to compute $F(\alpha)$ (in finite step), it suffices to know finite information $\bar{\alpha}(m)$ for some $m$.

Therefore $F$ must be continuous:

$$\forall \alpha \exists m \forall \beta \in \bar{\alpha}(m) (F(\alpha) = F(\beta)).$$

Combining this with an axiom of choice

$$\forall \alpha \exists n A(\alpha, n) \rightarrow \exists F \in \mathbb{N}^{\mathbb{N}^\mathbb{N}} \forall \alpha A(\alpha, F(\alpha))$$

we have WC-$\mathbb{N}$:

$$\forall \alpha \exists n A(\alpha, n) \rightarrow \forall \alpha \exists mn \forall \beta \in \bar{\alpha}(m) A(\beta, n).$$
Axioms of intuitionism (fan theorem)

Suppose that $A$ is quantifier free and upward closed, i.e.

$$A(a) \land a \preceq b \rightarrow A(b).$$

Let $a \in T \iff \neg A(a)$. Then $T$ is a tree, i.e.

$$b \in T \land a \preceq b \rightarrow a \in T.$$
Axioms of intuitionism (fan theorem)

The weak König lemma (WKL):

Every infinite binary tree has an infinite path is expressed by

\[ \forall n \exists a \in \{0, 1\}^* (|a| = n \land a \in T) \rightarrow \exists \alpha \in \{0, 1\}^N \forall n (\alpha(n) \in T). \]

It has a classical contraposition:

\[ \forall \alpha \in \{0, 1\}^N \exists n A(\alpha(n)) \rightarrow \exists n \forall a \in \{0, 1\}^* (|a| = n \rightarrow A(a)) \]

which is equivalent to FAN\textsubscript{D}:

\[ \forall \alpha \in \{0, 1\}^N \exists n A(\alpha(n)) \rightarrow \exists n \forall \alpha \in \{0, 1\}^N \exists k \leq n A(\alpha(k)). \]
Axioms of constructive recursive mathematics

- Extended Church’s thesis ($\text{ECT}_0$):

\[ \forall n [A(n) \rightarrow \exists m B(n, m)] \rightarrow \exists k \forall n [A(n) \rightarrow \exists m (T(k, n, m) \land B(n, U(m)))] \]

where $A$ is almost negative.

- Markov’s principle ($\text{MP}$, $\Sigma^0_1$-DNE):

\[ \neg \neg \exists n (\alpha(n) \neq 0) \rightarrow \exists n (\alpha(n) \neq 0) \]
Axioms of constructive recursive mathematics

The principle (Church’s thesis):

- each sequence of natural numbers is computable

is expressed by an axiom

\[ \forall \alpha \exists k \forall n \exists m \left( T(k, n, m) \land U(m) = \alpha(n) \right) \]

where \( T \) and \( U \) are Kleene’s \( T \)-predicate and the result-extracting function, respectively.
Axioms of constructive recursive mathematics

Combining this with an axiom of choice:
\[ \forall n \exists m B(n, m) \rightarrow \exists \alpha \forall n B(n, \alpha(n)) \]
we have the arithmetical form of Church’s thesis:
\[ \forall n \exists m B(n, m) \rightarrow \exists k \forall n \exists m [T(k, n, m) \land B(n, U(m))] \]
which has an extended form $\text{ECT}_0$:
\[ \forall n [A(n) \rightarrow \exists m B(n, m)] \rightarrow \exists k \forall n [A(n) \rightarrow \exists m (T(k, n, m) \land B(n, U(m)))] \]
where $A$ is almost negative.
Axioms of constructive mathematics

- The axiom of countable choice ($AC_0$):
  \[
  \forall n \exists y \in YA(n, y) \rightarrow \exists f \in Y^n \forall nA(n, f(n))
  \]

- The axiom of dependent choice ($DC$):
  \[
  \forall x \in X \exists y \in XA(x, y) \rightarrow \\
  \forall x \in X \exists f \in X^N[f(0) = x \land \forall nA(f(n), f(n + 1))]
  \]

- The axiom of unique choice ($AC!$):
  \[
  \forall x \in X \exists! y \in YA(x, y) \rightarrow \exists f \in Y^X \forall x \in XA(x, f(x))
  \]
Relationship between axioms

Proposition
ECT₀ + FAN₅D ⊨ ⊥ and ECT₀ + WC-N ⊨ ⊥.

Proposition
WKL ⊨ FAN₅D and FAN₅D ⊨ c WKL.

Proposition
ECT₀ + AC₁,₀ ⊨ ⊥, where AC₁,₀ is an axiom of choice:

∀α ∃nA(α, n) → ∃F ∈ N^{N,N} ∀αA(α, F(α))
Classical theorems on compactness:

- **MCT**: Every bounded monotone sequence of real numbers converges.
- **SCT**: Every sequence of a compact metric space has a convergent subsequence.
- **HBT**: Every (countable) open cover of a compact metric space has a finite subcover.
- **CIT**: Every family (sequence) of closed subsets of a compact metric space with finite intersection property has an intersection.
- **MIN**: Every uniformly continuous real function on a compact metric space attains its minimum.
Classical theorems on continuity:

- **DCT**: There exists a discontinuous mapping from $\mathbb{N}^\mathbb{N}$ into $\mathbb{N}$.
- **UCT**: Every continuous mapping from a compact metric space into a (separable) metric space is uniformly continuous.
Theorem (FAN) HBT.

Theorem (FAN)

If \( f : [0, 1] \rightarrow \mathbb{R} \) is a uniformly continuous function such that \( f(x) > 0 \) for each \( x \in [0, 1] \), then \( \inf(f) > 0 \).
Some mathematical consequences of axioms

Theorem (ECT₀)

ST: There exists a bounded monotone sequence \((q_n)_n\) of rational numbers such that for each \(x \in \mathbb{R}\)

\[ \exists m k \forall n \geq m (2^{-k} < |x - q_n|). \]

Theorem (ECT₀)

There exists a uniformly continuous function \(f : [0, 1] \to \mathbb{R}\) such that \(f(x) > 0\) for each \(x \in [0, 1]\) and \(\inf(f) = 0\).
Some mathematical consequences of axioms

Theorem (WC-N or ECT$_0$ + MP)

KLST: Every mapping from a complete separable metric space into a metric space is continuous.

Theorem (FAN)

UCT.

Theorem (ECT$_0$)

There exists a continuous mapping from $\{0, 1\}^\mathbb{N}$ into $\mathbb{N}$ which is not uniformly continuous.
The Friedman-Simpson-program (classical reverse mathematics) is
▶ a formal mathematics using classical logic,
▶ assuming a very weak set existence axiom.
▶ main question is ”Which set existence axioms are needed to prove the theorems of ordinary mathematics?”.  
▶ many theorems have been classified by set existence axioms of various strengths.
Classical reverse mathematics

The arithmetical comprehension axiom (ACA):

$$\exists X \forall n (n \in X \leftrightarrow A)$$

where $A$ is an arithmetical formula.

**Theorem**

*The following are equivalent.*

- ACA
- MCT
- SCT
Theorem

The following are equivalent.

- WKL
- HBT
- CIT
- MIN
- UCT
Constructive reverse mathematics (CRM)

Since classical reverse mathematics is formalized with classical logic, we cannot

- classify theorems in intuitionism nor in constructive recursive mathematics which are inconsistent with classical mathematics
- distinguish theorems from their contrapositions.
Constructive reverse mathematics (CRM)

Constructive mathematics is

- an informal mathematics using intuitionistic logic
- assuming some function existence axioms
- a core of the varieties of mathematics which can be extended to
  - intuitionism (by adding WC-N and FAN)
  - constructive recursive mathematics (by adding \( \text{ECT}_0 \) and MP)
  - classical mathematics (by adding PEM).
Constructive reverse mathematics (CRM)

The purpose of **constructive reverse mathematics** is

- to classify various theorems in intuitionistic, constructive recursive and classical mathematics
- by logical principles, function existence axioms and their combinations.
Logical principles

- The limited principle of omniscience (LPO, $\Sigma^0_1$-PEM):
  \[ \exists n (\alpha(n) \neq 0) \lor \neg \exists n (\alpha(n) \neq 0) \]

- The weak limited principle of omniscience (WLPO, $\Pi^0_1$-PEM):
  \[ \neg \exists n (\alpha(n) \neq 0) \lor \neg \neg \exists n (\alpha(n) \neq 0) \]
The lesser limited principle of omniscience (LLPO, $\Sigma^0_1$-DML):

$$\neg(\exists n(\alpha(n) \neq 0) \land \exists n(\beta(n) \neq 0)) \rightarrow$$

$$\neg\exists n(\alpha(n) \neq 0) \lor \neg\exists n(\beta(n) \neq 0)$$

Markov’s principle for disjunction (MP$^\vee$, $\Pi^0_1$-DML):

$$\neg(\neg\exists n(\alpha(n) \neq 0) \land \neg\exists n(\beta(n) \neq 0)) \rightarrow$$

$$\neg\neg\exists n(\alpha(n) \neq 0) \lor \neg\neg\exists n(\beta(n) \neq 0)$$
Logical principles

Proposition
$LPO \Rightarrow WLPO \Rightarrow LLPO \Rightarrow MP^\vee$.

Proposition
$LPO \Rightarrow MP \Rightarrow MP^\vee$.

Proposition
$LPO \Leftrightarrow WLPO + MP$.

Proposition
$ECT_0 + LLPO \vdash \bot$ and $WC-N + LLPO \vdash \bot$. 
Logical principles

- Weak Markov’s principle (WMP):

\[
\forall \beta [\neg \neg \exists n (\beta(n) \neq 0) \lor \neg \neg \exists n (\beta(n) \neq \alpha(n))] \rightarrow \exists n (\alpha(n) \neq 0)
\]

Proposition

MP ⇔ WMP + MP∧.

Proposition

ECT₀ ⊢ WMP and WC-N ⊢ WMP.
CRM with countable choice (compactness)

Definition
A metric space is **compact** if it is complete and totally bounded.

Theorem
*The following are equivalent.*
- LPO
- MCT
- SCT
CRM with countable choice (compactness)

Theorem

The following are equivalent.

- LLPO
- WKL
- CIT
- MIN
CRM with countable choice (compactness)

Definition

- A real function $f$ on a metric space has at most one minimum if $0 < d(x, y)$ implies $f(z) < f(x) \lor f(z) < f(y)$ for some $z$.
- A binary tree $T$ has at most one path if $\alpha \neq \beta$ implies $\overline{\alpha}(n) \notin T \lor \overline{\beta}(n) \notin T$ for some $n$.

Theorem

The following are equivalent.

- $\text{FAN}_D$
- Every infinite binary tree with at most one path has an infinite path.
- Every uniformly continuous real function on a compact metric space with at most one minimum attains its minimum.
Definition
A mapping \( f \) between metric spaces is **discontinuous** if there exist \( \delta > 0 \) and a sequence \( (x_n)_n \) converging to a limit \( x \) such that \( d(f(x_n), f(x)) \geq \delta \) for all \( n \).

Theorem
*The following are equivalent.*

- WLPO
- DCT
Definition
A mapping $f$ between metric spaces is nondiscontinuous if $x_n \rightarrow x$ and $d(f(x_n), f(x)) \geq \delta$ imply $\delta \leq 0$.

Theorem
The following are equivalent.

$\neg$WLPO

Every mapping from a complete metric space into a metric space is nondiscontinuous.
**Definition**
A mapping $f$ between metric spaces is strongly extensional if $0 < d(f(x), f(y))$ implies $0 < d(x, y)$.

**Theorem**
*The following are equivalent.*

- MP
- *Every mapping between metric space is strongly extensional.*
Definition
A mapping $f$ between metric spaces is **sequentially continuous** if $x_n \to x$ implies $f(x_n) \to f(x)$.

Theorem
The following are equivalent.

- **WMP**
- Every mapping from a complete metric space into a metric space is strongly extensional.
- Every nondiscontinuous mapping from a complete metric space into a metric space is sequentially continuous.
Definition
A subset $S$ of $\mathbb{N}$ is **pseudobounded** if $\lim_{n \to \infty} s_n/n = 0$ for each sequence $(s_n)_n$ in $S$.

Theorem
The following are equivalent.

- **BD-N**: Every countable pseudobounded subset is bounded.
- Every sequentially continuous mapping from a separable metric space into a metric space is continuous.

Proposition
$\text{ECT}_0 + \text{MP} \vdash \text{BD-N}$ and $\text{WC-N} \vdash \text{BD-N}$.
The following are equivalent.

- \(\neg\text{WLPO} + \text{WMP} + \text{BD-N}\)
- \(\text{KLST}\)
CRM without countable choice

- The axiom of countable number choice ($\text{AC}_{00}$):

\[
\forall m \exists n A(m, n) \rightarrow \exists \alpha \forall m A(m, \alpha(m))
\]

**Theorem**

*The following are equivalent.*

- LPO + $\Pi^0_1$-AC$_{00}$
- MCT
- SCT
CRM without countable choice

Definition

- A sequence \((F_n)_n\) of closed subsets of \(\{0, 1\}^\mathbb{N}\) is given by \(\sigma : \{0, 1\}^* \times \mathbb{N} \to \mathbb{N}\) such that
  \[\alpha \in F_n \iff \forall k (\sigma(\alpha(k), n) = 0).\]

- A sequence \((F_n)_n\) has finite intersection property if for each \(n\)
  \[\exists \alpha (\alpha \in \bigcap_{i=0}^{n} F_i).\]
CRM without countable choice

- The axiom of countable choice for disjunction ($AC_0^\lor$):

$$\forall m(A(m) \lor B(m)) \rightarrow \exists \alpha \in \{0, 1\}^N \forall m[(\alpha(m) = 0 \rightarrow A(m)) \land (\alpha(m) = 1 \rightarrow B(m))]$$

**Theorem**

*The following are equivalent.*

- $LLPO + \Pi^0_1-AC_0^\lor$
- $WKL$
- $CIT$: *Every sequence of closed subsets of $\{0, 1\}^N$ with finite intersection property has an intersection.*
CRM without countable choice

Definition

- A sequence \((G_n)_n\) of open subsets of \(\{0, 1\}^\mathbb{N}\) is given by

\[
\sigma : \{0, 1\}^* \times \mathbb{N} \to \mathbb{N}\n\]
such that

\[
\alpha \in G_n \iff \exists k(\sigma(\alpha(k), n) \neq 0).
\]

- A sequence \((G_n)_n\) is a **cover** if

\[
\forall \alpha \in \{0, 1\}^\mathbb{N}(\alpha \in \bigcup_n G_n).
\]
CRM without countable choice

Definition

A mapping $F : \{0, 1\}^\mathbb{N} \rightarrow \mathbb{N}$ is representable if there exists $\gamma : \{0, 1\}^* \rightarrow \mathbb{N}$ such that

$$\forall \alpha \in \{0, 1\}^\mathbb{N} \exists n[\forall k < n(\gamma(\overline{\alpha}(k)) = 0) \land \gamma(\overline{\alpha}(n)) = F(\alpha) + 1].$$

Theorem

The following are equivalent.

- FAN$_D$
- HBT: Every open cover has a finite subcover.
- UCT: Every representable mapping is uniformly continuous.