Constructive reverse mathematics: an introduction

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Contents

A history of constructivism

- History
  - Arithmetization of mathematics (Kronecker, 1987)
  - Three kinds of intuition (Poincaré, 1905)
  - French semi-intuitionism (Borel, 1914)
  - Intuitionism (Brouwer, 1914)
  - Predicativity (Weyl, 1918)
  - Finitism (Skolem, 1923; Hilbert-Bernays, 1934)
  - Constructive recursive mathematics (Markov, 1954)
  - Constructive mathematics (Bishop, 1967)
- Logic
  - Intuitionistic logic (Heyting, 1934; Kolmogorov, 1932)

Mathematical theory

A mathematical theory consists of

- axioms describing mathematical objects in the theory
- logic being used to derive theorems from the axioms

Two ways of generalizing mathematical theory:

axioms	logic
commutative group	classical logic
group	intuitionistic logic
monoid	minimal logic

Language

We use the standard language of (many-sorted) first-order predicate logic based on

- primitive logical operators $\wedge, \vee, \rightarrow, \bot, \forall, \exists$.

We introduce the abbreviations

- $\neg A \equiv A \rightarrow \bot$;
- $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$.

Deduction

We shall use $\mathcal{D}$, possibly with a subscript, for arbitrary deduction.

We write

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ A\end{array}$$

to indicate that $\mathcal{D}$ is deduction with conclusion A and assumptions Γ.

Deduction (Basis)

For each formula A,

$$A$$

is a deduction with conclusion A and assumptions $\{A\}$.

Deduction (Induction step)

If

$$\begin{array}{c}\Gamma\\ \mathcal{D}\\ B\end{array}$$

is a deduction, then

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{D}\\ B\end{array}}{A \to B}\ {\to}\mathrm{I}$$

is a deduction with conclusion $A \to B$ and assumptions $\Gamma \setminus \{A\}$.
We write

$$\dfrac{\begin{array}{c}[A]\\ \mathcal{D}\\ B\end{array}}{A \to B}\ {\to}\mathrm{I}$$

Deduction (Induction step)

If

$$\begin{array}{cc} \Gamma_1 & \Gamma_2 \\ \mathcal{D}_1 & \mathcal{D}_2 \\ A \to B & A \end{array}$$

are deductions, then

$$\dfrac{\begin{array}{c} \Gamma_1 \\ \mathcal{D}_1 \\ A \to B \end{array} \quad \begin{array}{c} \Gamma_2 \\ \mathcal{D}_2 \\ A \end{array}}{B} \to\mathrm{E}$$

is a deduction with conclusion B and assumptions $\Gamma_1 \cup \Gamma_2$.

Deduction (Induction step)

If

$$\begin{array}{ccc} \Gamma_1 & \Gamma_2 & \Gamma_3 \\ \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \\ A \vee B & C & C \end{array}$$

are deductions, then

$$\dfrac{\begin{array}{ccc} \Gamma_1 & \Gamma_2 & \Gamma_3 \\ \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \\ A \vee B & C & C \end{array}}{C}\ \vee\mathrm{E}$$

is a deduction with conclusion C and assumptions $\Gamma_1 \cup (\Gamma_2 \setminus \{A\}) \cup (\Gamma_3 \setminus \{B\})$.
We write

$$\dfrac{\begin{array}{ccc} & [A] & [B] \\ \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 \\ A \vee B & C & C \end{array}}{C}\ \vee\mathrm{E}$$

Minimal logic

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ B\end{array}}{A \wedge B}\ \wedge\mathrm{I} \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\end{array}}{A}\ \wedge\mathrm{E}_r \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\end{array}}{B}\ \wedge\mathrm{E}_l$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ A\end{array}}{A \vee B}\ \vee\mathrm{I}_r \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ B\end{array}}{A \vee B}\ \vee\mathrm{I}_l \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A \vee B\end{array} \quad \begin{array}{c}[A]\\ \mathcal{D}_2\\ C\end{array} \quad \begin{array}{c}[B]\\ \mathcal{D}_3\\ C\end{array}}{C}\ \vee\mathrm{E}$$

$$\dfrac{\begin{array}{c}[A]\\ \mathcal{D}\\ B\end{array}}{A \to B}\ \to\mathrm{I} \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ A \to B\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ A\end{array}}{B}\ \to\mathrm{E}$$

Minimal logic

$$\dfrac{\begin{array}{c}\mathcal{D}\\ A\end{array}}{\forall y A[x/y]}\ \forall\mathrm{I} \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \forall x A\end{array}}{A[x/t]}\ \forall\mathrm{E}$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ A[x/t]\end{array}}{\exists x A}\ \exists\mathrm{I} \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \exists y A[x/y]\end{array} \quad \begin{array}{c}[A]\\ \mathcal{D}_2\\ C\end{array}}{C}\ \exists\mathrm{E}$$

- In $\forall\mathrm{E}$ and $\exists\mathrm{I}$, t must be free for x in A.
- In $\forall\mathrm{I}$, $\mathcal{D}$ must not contain assumptions containing x free, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$.
- In $\exists\mathrm{E}$, $\mathcal{D}_2$ must not contain assumptions containing x free except A, $x \notin \mathrm{FV}(C)$, and $y \equiv x$ or $y \notin \mathrm{FV}(A)$.

Minimal logic

We denote by

$$\Gamma \vdash_m A$$

that there is a deduction in minimal logic with the conclusion A and the assumptions in Γ.

Example (minimal logic)

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\neg\neg(A \to B)] \qquad \dfrac{\dfrac{\dfrac{[\neg\neg A] \qquad \dfrac{\dfrac{[\neg B] \qquad \dfrac{[A \to B] \quad [A]}{B}\,{\to}\mathrm{E}}{\bot}\,{\to}\mathrm{E}}{\neg A}\,{\to}\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg(A \to B)}\,{\to}\mathrm{I}}{\bot}\,{\to}\mathrm{E}}{\neg\neg B}\,{\to}\mathrm{I}}{\neg\neg A \to \neg\neg B}\,{\to}\mathrm{I}}{\neg\neg(A \to B) \to (\neg\neg A \to \neg\neg B)}\,{\to}\mathrm{I}
$$

Intuitionistc logic

Intuitionistic logic is obtained from minimal logic by adding the intuitionistic absurdity rule:

$$\begin{array}{c} \mathcal{D} \\ \dfrac{\bot}{A}\ \bot_i \end{array}$$

We denote by

$$\Gamma \vdash_i A$$

that there is a deduction in intuitionistic logic with the conclusion A and the assumptions in Γ.
Note that

$$\Gamma \vdash_m A \Rightarrow \Gamma \vdash_i A.$$

Example (intuitionistic logic)

$$
\dfrac{\dfrac{\dfrac{[\neg\neg A \to \neg\neg B] \qquad \dfrac{\dfrac{[\neg(A \to B)] \qquad \dfrac{\dfrac{[\neg A] \quad [A]}{\bot}\,\bot_i}{A \to B}}{\bot}}{\neg\neg A}}{\neg\neg B} \qquad \dfrac{\dfrac{[\neg(A \to B)] \quad \dfrac{[B]}{A \to B}}{\bot}}{\neg B}}{\bot}}{\neg\neg(A \to B)}}{(\neg\neg A \to \neg\neg B) \to \neg\neg(A \to B)}
$$

Classical logic

Classical logic is obtained from intuitionistic logic by strengthening the absurdity rule to the classical absurdity rule:

$$\begin{array}{c} [\neg A] \\ \mathcal{D} \\ \dfrac{\bot}{A}\ \bot_c \end{array}$$

We denote by

$$\Gamma \vdash_c A$$

that there is a deduction in classical logic with the conclusion A and the assumptions in Γ.
Note that

$$\Gamma \vdash_i A \Rightarrow \Gamma \vdash_c A.$$

Example (classical logic)

The double negation elimination (DNE):

$$\dfrac{\dfrac{\dfrac{[\neg\neg A] \quad [\neg A]}{\bot}\,{\to}\mathrm{E}}{A}\,\bot_c}{\neg\neg A \to A}\,{\to}\mathrm{I}$$

Example (classical logic)

The principle of excluded middle (PEM):

$$\dfrac{\dfrac{[\neg(A \vee \neg A)] \qquad \dfrac{\dfrac{\dfrac{[\neg(A \vee \neg A)] \qquad \dfrac{[A]}{A \vee \neg A}\,\vee\mathrm{I}_r}{\bot}\,\to\mathrm{E}}{\neg A}\,\to\mathrm{I}}{A \vee \neg A}\,\vee\mathrm{I}_l}{\bot}\,\to\mathrm{E}}{A \vee \neg A}\,\bot_c$$

Example (classical logic)

De Morgan's law (DML):

$$
\dfrac{\dfrac{\dfrac{[\neg(\neg A \vee \neg B)] \qquad \dfrac{\dfrac{\dfrac{[\neg(A \wedge B)] \qquad \dfrac{[A] \quad [B]}{A \wedge B}\,\wedge\mathrm{I}}{\bot}\,\to\mathrm{E}}{\neg A}\,\to\mathrm{I}}{\neg A \vee \neg B}\,\vee\mathrm{I}_r}{\bot}\,\to\mathrm{E}}{\neg B}\,\to\mathrm{I}}{\neg A \vee \neg B}\,\vee\mathrm{I}_l \qquad [\neg(\neg A \vee \neg B)]}{\bot}\,\to\mathrm{E}}{\neg A \vee \neg B}\,\bot_c}{\neg(A \wedge B) \to \neg A \vee \neg B}\,\to\mathrm{I}
$$

Logical equivalences as equivalence relations

Each logic defines an equivalence relation $\sim_*$ on the set of formulas by

$$A \sim_* B \Leftrightarrow \vdash_* A \leftrightarrow B$$

where $* \in \{m, i, c\}$. We have

$$A \sim_m B \Rightarrow A \sim_i B \Rightarrow A \sim_c B$$

and

- $\neg\neg\neg A \in [\neg A]_m$
- $\neg\neg A \to \neg\neg B \notin [\neg\neg(A \to B)]_m$
- $\neg\neg A \to \neg\neg B \in [\neg\neg(A \to B)]_i$
- $\neg(A \wedge B) \notin [\neg A \vee \neg B]_i$
- $\neg(A \wedge B) \in [\neg A \vee \neg B]_c$

Notations

- $m, n, i, j, k, \ldots \in \mathbf{N}$
- $a, b, c, \ldots \in \mathbf{N}^*$
  - $|a| =$ the length of a
  - $a * b =$ the concatenation of a and b
  - $a \preceq b \Leftrightarrow \exists c(a * c = b)$
- $\alpha, \beta, \gamma, \delta, \ldots \in \mathbf{N}^{\mathbf{N}}$
  - $\overline{\alpha}(n) = (\alpha(0), \ldots, \alpha(n-1))$
  - $\alpha \in a \Leftrightarrow \overline{\alpha}(|a|) = a$
- $\{0,1\}^* = \{a \mid \forall n < |a|(a(n) = 0 \vee a(n) = 1)\}$
- $\{0,1\}^{\mathbf{N}} = \{\alpha \mid \forall n(\alpha(n) = 0 \vee \alpha(n) = 1)\}$

Axioms of intuitionism

- The weak continuity for numbers ($\mathrm{WC\text{-}N}$):

$$\forall\alpha\exists n A(\alpha, n) \to \forall\alpha\exists mn\forall\beta \in \overline{\alpha}(m) A(\beta, n)$$

- The fan theorem (FAN):

$$\forall\alpha \in \{0,1\}^{\mathbf{N}}\exists n A(\overline{\alpha}(n)) \to \exists n\forall\alpha \in \{0,1\}^{\mathbf{N}}\exists k \leq n A(\overline{\alpha}(k))$$

We denote FAN for quantifier free A by $\mathrm{FAN_D}$.

Axioms of intuitionism (continuity principle)

Let F be a function from $\mathbf{N}^{\mathbf{N}}$ into $\mathbf{N}$. Then

- to compute $F(\alpha)$ (in finite step), it suffices to know finite information $\overline{\alpha}(m)$ for some m.

Therefore F must be continuous:

$$\forall \alpha \exists m \forall \beta \in \overline{\alpha}(m)(F(\alpha) = F(\beta)).$$

Combining this with an axiom of choice

$$\forall \alpha \exists n A(\alpha, n) \to \exists F \in \mathbf{N}^{\mathbf{N}^{\mathbf{N}}} \forall \alpha A(\alpha, F(\alpha))$$

we have $\mathrm{WC\text{-}N}$:

$$\forall \alpha \exists n A(\alpha, n) \to \forall \alpha \exists m n \forall \beta \in \overline{\alpha}(m) A(\beta, n).$$

Axioms of intuitionism (fan theorem)

Suppose that A is quantifier free and upward closed, i.e.

$$A(a) \wedge a \preceq b \rightarrow A(b).$$

Let $a \in T \Leftrightarrow \neg A(a)$. Then T is a tree, i.e.

$$b \in T \wedge a \preceq b \rightarrow a \in T.$$

Axioms of intuitionism (fan theorem)

The weak König lemma (WKL):

- Every infinite binary tree has an infinite path

is expressed by

$$\forall n \exists a \in \{0,1\}^*(|a| = n \wedge a \in T) \to \exists \alpha \in \{0,1\}^{\mathbf{N}} \forall n(\overline{\alpha}(n) \in T).$$

It has a classical contraposition:

$$\forall \alpha \in \{0,1\}^{\mathbf{N}} \exists n A(\overline{\alpha}(n)) \to \exists n \forall a \in \{0,1\}^*(|a| = n \to A(a))$$

which is equivalent to $\mathrm{FAN}_{\mathrm{D}}$:

$$\forall \alpha \in \{0,1\}^{\mathbf{N}} \exists n A(\overline{\alpha}(n)) \to \exists n \forall \alpha \in \{0,1\}^{\mathbf{N}} \exists k \leq n A(\overline{\alpha}(k)).$$

Axioms of constructive recursive mathematics

- Extended Church's thesis (ECT_0):

$$\forall n[A(n) \to \exists m B(n,m)] \to \\ \exists k \forall n[A(n) \to \exists m(T(k,n,m) \wedge B(n,U(m)))]$$

  where A is almost negative.

- Markov's principle (MP, Σ^0_1-DNE):

$$\neg\neg\exists n(\alpha(n) \neq 0) \to \exists n(\alpha(n) \neq 0)$$

Axioms of constructive recursive mathematics

The principle (Church's thesis):

- each sequence of natural numbers is computable

is expressed by an axiom

$$\forall \alpha \exists k \forall n \exists m (T(k, n, m) \wedge U(m) = \alpha(n))$$

where T and U are Kleene's T-predicate and the result-extracting function, respectively.

Axioms of constructive recursive mathematics

Combining this with an axiom of choice:

$$\forall n \exists m B(n, m) \to \exists \alpha \forall n B(n, \alpha(n))$$

we have the arithmetical form of Church's thesis:

$$\forall n \exists m B(n, m) \to \exists k \forall n \exists m [T(k, n, m) \wedge B(n, U(m))]$$

which has an extended form ECT_0:

$$\forall n [A(n) \to \exists m B(n, m)] \to \\ \exists k \forall n [A(n) \to \exists m (T(k, n, m) \wedge B(n, U(m)))]$$

where A is almost negative.

Axioms of constructive mathematics

- The axiom of countable choice (AC_0):

$$\forall n \exists y \in Y A(n, y) \to \exists f \in Y^{\mathbf{N}} \forall n A(n, f(n))$$

- The axiom of dependent choice (DC):

$$\forall x \in X \exists y \in X A(x, y) \to$$
$$\forall x \in X \exists f \in X^{\mathbf{N}} [f(0) = x \wedge \forall n A(f(n), f(n+1))]$$

- The axiom of unique choice ($\mathrm{AC!}$):

$$\forall x \in X \exists! y \in Y A(x, y) \to \exists f \in Y^X \forall x \in X A(x, f(x)).$$

Relationship between axioms

Proposition

$\mathrm{ECT}_0 + \mathrm{FAN}_\mathrm{D} \vdash \bot$ *and* $\mathrm{ECT}_0 + \mathrm{WC\text{-}N} \vdash \bot$.

Proposition

$\mathrm{WKL} \vdash \mathrm{FAN}_\mathrm{D}$ *and* $\mathrm{FAN}_\mathrm{D} \vdash_c \mathrm{WKL}$.

Proposition

$\mathrm{ECT}_0 + \mathrm{AC}_{1,0} \vdash \bot$, *where* $\mathrm{AC}_{1,0}$ *is an axiom of choice:*

$$\forall \alpha \exists n A(\alpha, n) \to \exists F \in \mathbf{N}^{\mathbf{N}^{\mathbf{N}}} \forall \alpha A(\alpha, F(\alpha))$$

Classical theorems

Classical theorems on compactness:

- MCT: Every bounded monotone sequence of real numbers converges.
- SCT: Every sequence of a compact metric space has a convergent subsequence.
- HBT: Every (countable) open cover of a compact metric space has a finite subcover.
- CIT: Every family (sequence) of closed subsets of a compact metric space with finite intersection property has an intersection.
- MIN: Every uniformly continuous real function on a compact metric space attains its minimum.

Classical theorems

Classical theorems on continuity:

- DCT: There exists a discontinuous mapping from $\mathbf{N}^{\mathbf{N}}$ into $\mathbf{N}$.
- UCT: Every continuous mapping from a compact metric space into a (separable) metric space is uniformly continuous.

Some mathematical consequences of axioms

Theorem (FAN)

HBT.

Theorem (FAN)

If $f : [0,1] \to \mathbf{R}$ *is a uniformly continuous function such that* $f(x) > 0$ *for each* $x \in [0,1]$*, then* $\inf(f) > 0$.

Some mathematical consequences of axioms

Theorem (ECT_0)

ST: *There exists a bounded monotone sequence $(q_n)_n$ of rational numbers such that for each $x \in \mathbf{R}$*

$$\exists mk \forall n \geq m(2^{-k} < |x - q_n|).$$

Theorem (ECT_0)

There exists a uniformly continuous function $f : [0,1] \to \mathbf{R}$ such that $f(x) > 0$ for each $x \in [0,1]$ and $\inf(f) = 0$.

Some mathematical consequences of axioms

Theorem ($\mathrm{WC\text{-}N}$ or $\mathrm{ECT}_0 + \mathrm{MP}$)

KLST: *Every mapping from a complete separable metric space into a metric space is continuous.*

Theorem (FAN)

UCT.

Theorem (ECT_0)

There exists a continuous mapping from $\{0,1\}^{\mathbf{N}}$ *into* $\mathbf{N}$ *which is not uniformly continuous.*

Classical reverse mathematics

The Friedman-Simpson-program (classical reverse mathematics) is

- a formal mathematics using classical logic,
- assuming a very weak set existence axiom.
- main question is "Which set existence axioms are needed to prove the theorems of ordinary mathematics?".
- many theorems have been classified by set existence axioms of various strengths.

Classical reverse mathematics

The arithmetical comprehension axiom (ACA):

$$\exists X \forall n (n \in X \leftrightarrow A)$$

where A is an arithmetical formula.

Theorem

The following are equivalent.

- ACA
- MCT
- SCT

Classical reverse mathematics

Theorem

The following are equivalent.

- WKL
- HBT
- CIT
- MIN
- UCT

Constructive reverse mathematics (CRM)

Since classical reverse mathematics is formalized with classical logic, we cannot

- classify theorems in intuitionism nor in constructive recursive mathematics which are inconsistent with classical mathematics
- distinguish theorems from their contrapositions.

Constructive reverse mathematics (CRM)

Constructive mathematics is

- an informal mathematics using intuitionistic logic
- assuming some function existence axioms
- a core of the varieties of mathematics which can be extended to
  - intuitionism (by adding $\mathrm{WC\text{-}N}$ and FAN)
  - constructive recursive mathematics (by adding ECT_0 and MP)
  - classical mathematics (by adding PEM).

Constructive reverse mathematics (CRM)

The purpose of constructive reverse mathematics is

- to classify various theorems in intuitionistic, constructive recursive and classical mathematics
- by logical principles, function existence axioms and their combinations.

Logical principles

- The limited principle of omniscience (LPO, Σ^0_1-PEM):

$$\exists n(\alpha(n) \neq 0) \vee \neg\exists n(\alpha(n) \neq 0)$$

- The weak limited principle of omniscience (WLPO, Π^0_1-PEM):

$$\neg\exists n(\alpha(n) \neq 0) \vee \neg\neg\exists n(\alpha(n) \neq 0)$$

Logical principles

- The lesser limited principle of omnicience (LLPO, Σ^0_1-DML):

$$\neg(\exists n(\alpha(n) \neq 0) \wedge \exists n(\beta(n) \neq 0)) \rightarrow \\ \neg\exists n(\alpha(n) \neq 0) \vee \neg\exists n(\beta(n) \neq 0)$$

- Markov's principle for disjunction ($\mathrm{MP}^{\vee}$, Π^0_1-DML):

$$\neg(\neg\exists n(\alpha(n) \neq 0) \wedge \neg\exists n(\beta(n) \neq 0)) \rightarrow \\ \neg\neg\exists n(\alpha(n) \neq 0) \vee \neg\neg\exists n(\beta(n) \neq 0)$$

Logical principles

Proposition

$\mathrm{LPO} \Rightarrow \mathrm{WLPO} \Rightarrow \mathrm{LLPO} \Rightarrow \mathrm{MP}^{\vee}$.

Proposition

$\mathrm{LPO} \Rightarrow \mathrm{MP} \Rightarrow \mathrm{MP}^{\vee}$.

Proposition

$\mathrm{LPO} \Leftrightarrow \mathrm{WLPO} + \mathrm{MP}$.

Proposition

$\mathrm{ECT}_0 + \mathrm{LLPO} \vdash \bot$ *and* $\mathrm{WC\text{-}N} + \mathrm{LLPO} \vdash \bot$.

Logical principles

- Weak Markov's principle (WMP):

$$\forall\beta[\neg\neg\exists n(\beta(n) \neq 0) \vee \neg\neg\exists n(\beta(n) \neq \alpha(n))] \to \exists n(\alpha(n) \neq 0)$$

Proposition

$\mathrm{MP} \Leftrightarrow \mathrm{WMP} + \mathrm{MP}^{\vee}$.

Proposition

$\mathrm{ECT}_0 \vdash \mathrm{WMP}$ *and* $\mathrm{WC\text{-}N} \vdash \mathrm{WMP}$.

CRM with countable choice (compactness)

Definition

A metric space is compact if it is complete and totally bounded.

Theorem

The following are equivalent.

- LPO
- MCT
- SCT

CRM with countable choice (compactness)

Theorem

The following are equivalent.

- LLPO
- WKL
- CIT
- MIN

CRM with countable choice (compactness)

Definition

- A real function f on a metric space has at most one minimum if $0 < d(x, y)$ implies $f(z) < f(x) \vee f(z) < f(y)$ for some z.
- A binary tree T has at most one path if $\alpha \neq \beta$ implies $\overline{\alpha}(n) \notin T \vee \overline{\beta}(n) \notin T$ for some n.

Theorem

The following are equivalent.

- $\mathrm{FAN}_{\mathrm{D}}$
- *Every infinite binary tree with at most one path has an infinite path.*
- *Every uniformly continuous real function on a compact metric space with at most one minimum attains its minimum.*

CRM with countable choice (continuity)

Definition

A mapping f between metric spaces is discontinuous if there exist $\delta > 0$ and a sequence $(x_n)_n$ converging to a limit x such that $d(f(x_n), f(x)) \geq \delta$ for all n.

Theorem

The following are equivalent.

- WLPO
- DCT

CRM with countable choice (continuity)

Definition

A mapping f between metric spaces is nondiscontinuous if $x_n \to x$ and $d(f(x_n), f(x)) \geq \delta$ imply $\delta \leq 0$.

Theorem

The following are equivalent.

- $\neg\mathrm{WLPO}$
- *Every mapping from a complete metric space into a metric space is nondiscontinuous.*

CRM with countable choice (continuity)

Definition

A mapping f between metric spaces is strongly extensional if $0 < d(f(x), f(y))$ implies $0 < d(x, y)$.

Theorem

The following are equivalent.

- MP
- *Every mapping between metric space is strongly extensional.*

CRM with countable choice (continuity)

Definition

A mapping f between metric spaces is sequentially continuous if $x_n \to x$ implies $f(x_n) \to f(x)$.

Theorem

The following are equivalent.

- WMP
- *Every mapping from a complete metric space into a metric space is strongly extensional.*
- *Every nondiscontinuous mapping from a complete metric space into a metric space is sequentially continuous.*

CRM with countable choice (continuity)

Definition

A subset S of $\mathbf{N}$ is pseudobounded if $\lim_{n\to\infty} s_n/n = 0$ for each sequence $(s_n)_n$ in S.

Theorem

The following are equivalent.

- BD-N: *Every countable pseudobounded subset is bounded.*
- *Every sequentially continuous mapping from a separable metric space into a metric space is continuous.*

Proposition

$\mathrm{ECT}_0 + \mathrm{MP} \vdash \mathrm{BD\text{-}N}$ *and* $\mathrm{WC\text{-}N} \vdash \mathrm{BD\text{-}N}$.

CRM with countable choice (continuity)

Theorem

The following are equivalent.

- $\neg\mathrm{WLPO} + \mathrm{WMP} + \mathrm{BD\text{-}N}$
- KLST

CRM without countable choice

- The axiom of countable number choice (AC_{00}):

$$\forall m \exists n A(m,n) \to \exists \alpha \forall m A(m, \alpha(m))$$

Theorem

The following are equivalent.

- $\mathrm{LPO} + \Pi^0_1\text{-}\mathrm{AC}_{00}$
- MCT
- SCT

CRM without countable choice

Definition

- A sequence $(F_n)_n$ of closed subsets of $\{0,1\}^{\mathbf{N}}$ is given by $\sigma : \{0,1\}^* \times \mathbf{N} \to N$ such that

$$\alpha \in F_n \Leftrightarrow \forall k(\sigma(\overline{\alpha}(k), n) = 0.$$

- A sequence $(F_n)_n$ has finite intersection property if for each n

$$\exists \alpha(\alpha \in \bigcap_{i=0}^{n} F_i).$$

CRM without countable choice

- The axiom of countable choice for disjunction ($\mathrm{AC}_0^{\vee}$):

$$\forall m(A(m) \vee B(m)) \rightarrow \\ \exists \alpha \in \{0,1\}^{\mathbf{N}} \forall m[(\alpha(m) = 0 \rightarrow A(m)) \wedge (\alpha(m) = 1 \rightarrow B(m))]$$

Theorem

The followng are equivalent.

- $\mathrm{LLPO} + \Pi_1^0\text{-}\mathrm{AC}_0^{\vee}$
- WKL
- CIT: *Every sequence of closed subsets of* $\{0,1\}^{\mathbf{N}}$ *with finite intersection property has an intersection.*

CRM without countable choice

Definition

- A sequence $(G_n)_n$ of open subsets of $\{0,1\}^{\mathbf{N}}$ is given by $\sigma : \{0,1\}^* \times \mathbf{N} \to N$ such that

$$\alpha \in G_n \Leftrightarrow \exists k(\sigma(\overline{\alpha}(k), n) \neq 0.$$

- A sequence $(G_n)_n$ is a cover if

$$\forall \alpha \in \{0,1\}^{\mathbf{N}}(\alpha \in \bigcup_n G_n).$$

CRM without countable choice

Definition

- A mapping $F : \{0,1\}^{\mathbf{N}} \to \mathbf{N}$ is representable if there exists $\gamma : \{0,1\}^* \to \mathbf{N}$ such that

$$\forall \alpha \in \{0,1\}^{\mathbf{N}} \exists n [\forall k < n(\gamma(\overline{\alpha}(k)) = 0) \wedge \gamma(\overline{\alpha}(n)) = F(\alpha) + 1].$$

Theorem

The followng are equivalent.

- $\mathrm{FAN_D}$
- HBT: *Every open cover has a finite subcover.*
- UCT: *Every representable mapping is uniformly continuous.*