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INVITED TALKS

Constructive finite free resolutions

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Northcott in his book Finite Free Resolutions presents some key results of Buchsbaum and Eisenbud both in a simplified way and without Noetherian hypothesis (using Hochster's notion of latent nonzero divisor). Together with Henri Lombardi and Claude Quitte, we have simplified further the proofs of these results, removing all use of non effective objects such as prime ideals or minimal prime ideals, This follows a general method for interpreting such non effective objects. In particular we obtain an algorithm computing the gcd of elements, given a finite free resolution of the ideal generated by these elements, as a constructive proof that any regular ring is a gcd domain.

Constructive mathematics and philosophy of mathematics

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Recent years have seen the flourishing of constructive mathematics Bishop style. However, the mathematical steady progress has not been equated by a comparable abundance of philosophical reflection on this mathematical practice.

Constructive mathematics is often associated with either Brouwer's intuitionism or Dummett's constructivism. Such an association seems nevertheless prima facie problematic, as the philosophical justifications of Brouwer's intuitionism and Dummett's constructivism suggest a rejection tout court of classical mathematics, as either illegitimate or, strictly speaking, unintelligible. As Fred Richman has stressed, constructive mathematics is a generalization of classical mathematics, since it is based (only) on a weaker logic. Classical mathematics is therefore fully intelligible, and in fact often instigates the constructive practice, by suggesting new problems as well as, in some cases, possible routes to their solutions. What are then the philosophical options for constructive mathematics Bishop style? In this talk I shall propose some initial thoughts in this direction.

Continuity is overrated (. . . sometimes)

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A, for the uninitiated, bizarre feature of Brouwer's intuitionism is that all functions $f : [0,1] \to \mathbb{R}$ are continuous. A similar phenomenon is well known in recursive models of mathematics, where this is sometimes known as 'Čeitins theorem' (Russian school), 'Kreisel, Lacombe, and Shoenfield' theorem (Polish/Classical school), or simply runs under the slogan 'computability implies continuity'. The question whether all functions $f : [0,1] \to \mathbb{R}$ are continuous or not has been hotly debated by the philosophically minded ever since Brouwer. The solomonic and fruitful solution adapted by Bishop was to simply ignore the issue and just prove things about (uniformly) continuos functions without taking a stance on whether there are or are not discontinuous functions. There are, however, a few constructive results about functions requiring weaker assumptions than continuity. Among them, for example:

Proposition 1 (Ishihara's second trick)**.** *Let f be a strongly extensional mapping of a complete metric space X into a metric space Y, and let $(x_n)_{n\geqslant 1}$ be a sequence in X converging to a limit x. Then for all positive numbers $\alpha < \beta$, either $\rho(f(x_n), f(x)) < \beta$ eventually or $\rho(f(x_n), f(x)) > \alpha$ infinitely often.*

Actually, this result is uninteresting classically, since it is a simple consequence of the law of excluded middle. It is just as uninteresting from Brouwer's or a recursive point of view, since there the first alternative is always true and the second alternative cannot happen. What is interesting, however, is that one is able to make this decision without knowing whether—vaguely speaking—one works in a classical universe or a intuitionistic/recursive one. Even more interesting about it is that constructively we can prove meaningful results about functions that are not necessarily continuous.

In this talk we are going to present a variety of similar results, showing that sometimes continuity is overrated.

Intuitionistic logic in categorial proof theory

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The characterization of logical constants by proof-theoretical means is related to Lawvere's discovery that intuitionistic logical constants are tied to functors in adjoint situations. This connection with the very important categorial notion of adjunction sheds much light on the normalization of proofs. The connection, which is rather good for intuitionistic logic ("rather good" and not "perfect" because of distribution of conjunction over disjunction), is not to be found to the same extent in classical logic, though that logic too can be understood sensibly in terms of categorial proof theory.

Reducibility method and logical relations in intuitionistic logic and programming languages[1]

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The strong normalisation of simply typed λ-calculus is the property that terms typeable in simply typed λ-calculus terminate. This fundamental property of type systems was first proved by Tait in [8] using the *reducibility method*. Later this method was used to prove strong normalization of polymorphic (second-order) lambda calculus, intersection type systems, calculus with explicit substitutions and various other type systems. It was also applied to characteise other reduction properties such as normalisation, head normalisation, weak head normalisation etc. Emerging from these proofs, the reducibility method became a widely accepted technique for proving various reduction properties of terms typeable in different type systems. The reader is referred to [1], a comprehensive reference book on type systems.

The main idea of this method is to relate terms typeable in a certain type system and terms satisfying certain reduction properties such as strong normalisation, head normalisation etc. It can be represented by a unary predicate $P_\alpha(t)$, which means that a term t typeable by α satisfies the property P. To this aim types are interpreted as suitable sets of terms called saturated or stable sets. Then, the soundness of type assignment is obtained with respect to these interpretations. A consequence of soundness is that every term typeable in the type system belongs to the interpretation of its type. This is an intermediate step between the terms typeable in the given type system and terms satisfying the considered property P. In general, the principal notions of the reducibility method are: 1. type interpretations (based on the considered property P); 2. term valuations; 3. closure conditions; 4. soundness of the type assignment. Suitable modifications of the reducibility method lead to uniform proofs of other reduction properties. An overview can be found in [3].

The basic relationship between logic and computation is given by the

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Curry-Howard correspondence [4] between simply typed λ-calculus and intuitionistic natural deduction. This connection extends to many other calculi and logical systems. We will give an overview of reducibility based proofs and relate them to logical counterparts.

In the setting of classical logic, the reducibility method is not well suited to prove strong normalization for $\lambda\mu$-calculus, the simply typed classical term calculus (see [2]). The symmetric candidates technique used to prove strong normalisation employs a fixed-point technique to define the reducibility candidates.

Logical relations were introduced by Statman in [7] to proof the confluence (the Church-Rosser property) of $\beta\eta$-reduction of the simply typed λ-terms. It was further developed by Mitchell in [5] and became a well-known method for proving confluence and standardisation in various type systems. Similarly to the reducibility method, the key notions are type interpretations. Logical relations in turn is a method based on binary relations $R_\alpha(t, t')$, which relate terms t and t' typeable by the type α that satisfy the relation R. Types are then interpreted as admissible relations.

In programming languages it is often necessary to relate terms either from the same language or from different languages in order to show their equivalence. To this aim logical relations became a powerful tool in programming languages (see Pierce [6]). We will point out some of the most important applications. Observational equivalence: logical relations prove that terms obtained by optimisation are equivalent. Compiler correctness: logical relations are employed to relate the source and target language. Security information flow: logical relations prove that the system prevents high security data to leak in low security output.

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Variants of forcing

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In [1] Lubarsky and Diener construct a group of related models to separate the four major varieties of Brouwer's fan theorem, over **IZF**, using uniform methods. Taking ideas from [1], we define semi-classical forcing models. These models generalize intuitionistic forcing models (the simplest Kripke models for **IZF**) by

(i) allowing weaker forcing semantics: making it in general easier to force a formula (the weakest semantics being classical forcing);

(ii) allowing us to restrict the universe of the model.

Semi-classical forcing models encompass both intuitionistic and classical forcing as well as the models of [1]. We believe that dealing directly with the forcing relation in this manner reduces the technical difficulties in the Lubarsky-Diener separating models and gives a better understanding of these models, in particular it gives a uniform basis for these models. The proof that semi-classical forcing models satisfy **IZF** is a straightforward adaption of the proof for intuitionistic forcing.

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Constructive reverse mathematics: an introduction

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We start with a quick review of a history of constructivism [9, 1.4] including Brouwer's intuitionism [5, 3], Markov's constructive recursive mathematics [7, 3] and Bishop's constructive mathematics [1, 4, 3, 4]. A mathematical theory consists of *axioms* describing mathematical objects in the theory and *logic* being used to derive theorems of the theory from the axioms. The logic used in those constructive mathematics is *intuitionistic logic* which is weaker than *classical logic* used in ordinal mathematics.

We highlight the difference between intuitionistic logic and classical logic; see [9, 2.1,2.3]. When we regard logical equivalence between propositions as an equivalence relation, the equivalence relation $\sim_i$ in intuitionistic logic is a subset of the equivalence relation $\sim_c$ in classical logic, and therefore the classification by $\sim_i$ is *finer* than the classification by $\sim_c$.

We then examine the axioms in Brouwer's intuitionism: the schema of weak continuity for numbers (WC-N) and the fan theorem (FAN), and the axioms in Markov's constructive recursive mathematics: the extended Church's thesis (ECT_0) and Markov's principle (MP); see [9, 4.3,4.5,4.6-7]. As their consequences, we see the Heine-Borel theorem (HBT) on compactness and the Kreisel-Lacombe-Shoenfield-Tsejtin theorem (KLST) on continuity in Brouwer's intuitionism, and the Specker theorem (ST) concerning compactness and KLST in Markov's constructive recursive mathematics; see [9, 6.3-4,7.2,7.4] Note that ST and KLST are *inconsistent* with ordinal mathematics.

The Friedman-Simpson (classical) reverse mathematics [8] aims at exploring, using classical logic, set existence axioms equivalent to various theorems in ordinal mathematics, and classifies theorems into a hierarchy (*linearly ordered structure*) of set existence axioms such as the weak König lemma (WKL), the arithmetical comprehension axiom (ACA) and so on, over the weak system $\mathbf{RCA}_0$ of second-order arithmetic whose objects are natural numbers and sets of natural numbers. For example, HBT is equivalent to WKL in classical reverse mathematics.

Since the equivalence relation $\sim_{\mathbf{RCA}_0}$ in classical reverse mathematics is based on classical logic, FAN, which is a contraposition of WKL, is classified into the equivalence class $[\mathrm{WKL}]_{\mathbf{RCA}_0}$, and ST and KLST are classified into the equivalence class $[\bot]_{\mathbf{RCA}_0}$ of absurdity $\bot$. This observation motivates our constructivisation of classical reverse mathematics, as intuitionistic logic enable us to have a finer classification.

Constructive reverse mathematics [6] aims at classifying theorems, using intuitionistic logic, into a *lattice structure* of nonconstructive logical principles such as instances of the principle of excluded middle (PEM), double negation elimination (DNE), De Morgan's law (DML) and so on, function (set) existence axioms such as instances of countable choice (CC), dependent choice (DC) and so on, and their combinations.

We present some results in constructive reverse mathematics *with* CC including those on HBT and the Cantor intersection theorem (CIT), and on KLST. We then compare the results on compactness with corresponding results in constructive reverse mathematics *without* CC, and highlight roles of instances of CC in constructive reverse mathematics.

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Unified approach to real numbers in various mathematical settings

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In the talk we discuss the fact that various classically equivalent definitions/constructions of the set of real numbers are not equivalent constructively, usually forcing us to make a setting-dependent choice and preventing us from directly applying results from one variety of constructivism to others.

The purpose of the talk is to resolve this problem and consists of the following.

- We present a definition of $\mathbb{R}$ which entails its three crucial structures (order, algebra, topology), clearly captures the intuition behind the real numbers and is setting-independent. The tool we use for this is the universal property from category theory.

  We define a *streak* to be a strict archimedean order, equipped with algebra which preserves this order (addition, as well as multiplication of positive elements) and in which $<$ is open and $\leq$ closed. A *streak morphism* is a map which preserves all this structure; we obtain a category. The usual number sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are all examples of streaks and it turns out that they can all be given via some universal property (determining them up to isomorphism). In particular, $\mathbb{R}$ can be defined as the terminal streak.

- We interpret this theory in various settings. We observe that all the usual constructions of reals in their corresponding settings satisfy our definition (Cauchy reals in settings with countable choice, Dedekind cuts in topoi, open Dedekind cuts in ASD, the formal space of reals in formal topology, the euclidean reals in classical topology...)

- We use streaks for an in-depth study of the structure of reals. It turns out that pieces of it correspond to reflections on the category of streaks. This means that we have canonical ways to transform streaks into ones with additional structure, and due to terminality these transformations are isomorphisms for $\mathbb{R}$. Hence we obtain for every concrete construction of $\mathbb{R}$ explicit formulae for all its structure.

If time permits, we will also discuss generalizations of this theory which allow us to characterize lower reals, upper reals or metric completions.

Three lectures on constructive algebra

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1. Structure of finitely generated abelian groups

The first lecture introduces some main features of constructive algebra w.r.t. classical algebra through the example of the structure theorem describing finitely generated abelian groups, and its generalization to finitely generated modules over a PID.

The true constructive content of these theorems is the Smith's algorithm for reducing matrices to a convenient normal form.

The theory of matrices over a ring appears also in the theory of solutions of linear systems with coefficients and unknowns in the given ring.

The notion of coherent ring (a ring in which finitely generated ideals are finitely presented) appears as the crucial notion for managing linear systems over a commutative ring.

In classical mathematics this notion is almost everywhere hidden behind the notion of Nœtherianity. We explore the interplay between coherence and Nœtherianity from the constructive point of view.

2. Constructive aspects of Krull dimension

Krull dimension is a central concept in commutative algebra. It appears in the hypotheses of many "great theorems".

Based on chains of prime ideals, the notion of Krull dimension of a ring should appear strongly as a nonconstructive one.

Nevertheless it is possible to give in classical mathematics an equivalent definition that uses only quantifications over elements of the ring (without speaking of prime ideals).

We explain why this new "elementary" definition of the Krull dimension allows us to manage in an algorithmic way many (and hopefully all) "great theorems" that use Krull dimension in classical mathematics.

This is a striking example showing that many classical theorems that appear a priori out of the scope of constructive mathematics can in fact be deciphered as hiding in their "abstract" proofs concrete algorithms for constructing the conclusion from the hypotheses, when they are conveniently reformulated in an elementary form (equivalent to the abstract form appearing in classical mathematics).

3. Dynamical Method in Constructive Algebra

In constructive analysis à la Bishop, or in certified numerical analysis, real numbers and continuous real functions are considered as given by finite rational approximations.

This corresponds to the Poincaré's program: "Ne jamais perdre de vue que toute proposition sur l'infini doit être la traduction, l'énoncé abrégé de propositions sur le fini." (Never lose sight of the fact that every proposition concerning infinity must be the translation, the precise statement of propositions concerning the finite.)

Abstract classical algebra uses also many kinds of infinite objects, but in general without finite approximations. Also, contrarily to the infinite objects of Analysis, these algebraic objects are often shown lacking completely of existence. E.g. the algebraic closure of a general field cannot be "constructed", contrarily to the completion of a general metric space.

The dynamical method considers that algebraic objects that are too abstract (too abstract, because they are impossible to construct) can nevertheless be considered through their finite approximations. Approximations of what? Approximations of more acurate compatible finite approximations! So, existence is replaced by formal compatibility, in the spirit of Hilbert's program.

E.g., the elementary definition of Krull dimension was obtained by considering finite approximations of chains of prime ideals.

In the talk we give some examples of this dynamical machinery: deciphering abstract proofs that use abstract infinite objects, and transforming these proofs in concrete algorithms dealing with finite approximations or the ideal objects.

We think that this is a very general method and that we have in hands the tools for realizing the Poincaré's program (an the Hilbert's programm) for abstract algebra.

This correponds to a logical framework called "Geometric theories".

Topological and Kripke models for constructivism

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Upon first learning about constructivism, a common reaction is to think, "How am I to understand how this could be true? How could Excluded Middle fail, for instance? Or reals be neither equal nor unequal to each other?" Often an explanation is given along the lines of an assertion being not about truth, but rather about knowledge, or provability, or computability. Rather than ask the listener to give up the belief that P is either true or false, it is argued that the assertion "P or not P" means that you either know P to be true or know P to be false, the failure of which is easily assented to. Similarly if one interprets the statement as that P is either provable or falsifiable, or that there is a computation of P or its negation. Those are all informal models of constructivism. They give a meaning, or semantics, to the notions of constructive logic and mathematics. If no more mathematical detail is offered, then these models remain on an informal level. The goal of this series is to present and work with some formal models. An added benefit is that we are able in the end to interpret assertions back as being about truth. The logic behind classical mathematics was worked out by Boole in the 19th century, and captured in the structure of a Boolean algebra. After Brouwer, the same was done for constructivism by Heyting, and summarized in the notion of a Heyting algebra. A Heyting-valued model is one which uses a Heyting algebra as the set of truth values. A particularly easy class of Heyting algebras to visualize and work with are the (collections of open sets of) topological spaces, which are called topological models. Certain topological spaces are of central importance to mathematics, such as the reals and the complex numbers, so it is no surprise that topological models over those spaces end up modeling interesting phenomena, which we will examine. Other models we will consider arise in contrast from principles of interest themselves, which lead us to develop new spaces to analyze. As it turns out, the theory of Heyting-valued models is closely related to forcing in set theory. In fact, forcing can be viewed as decomposable into a Heyting-valued model followed by a double-negation translation. Although this series cannot go much into forcing, the ideas behind this will be sketched. A special kind of topological space is an Alexandrov topology. This could be viewed as a partial order. When specializing to this case, such a model is called a Kripke model. One advantage of Kripke models is that they provide the simplest models for constructivism. For instance, to falsify Excluded Middle, consider a three-point partial order shaped like a "V". Making P true at one terminal node and false at the other, easily P is neither at the bottom. Another advantage is that it's easier with them to work with non-full models. The full model over a space consists of throwing in everything you possibly could; with a non-full model, you get to be selective about what's included. Some phenomena require a non-full model for their modeling, which is hence more easily done with a Kripke model. The goal of this series is both to provide the background theory as well as to exhibit lots of example models of interesting phenomena.

Why constructive topology must be point-free[1]

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There exist many different approaches to develop constructive mathematics (see for example [BR87]). There are also various foundations to formalize them formulated either in the style of an axiomatic set theory, as Aczel's CZF [Acz78, Acz82, Acz86] or Friedman's IZF [Bee85], or in category theory as topoi or pretopoi [MM92, JM95] or in type theory as Martin-Löf's type theory [NPS90] or Coquand's Calculus of Inductive Constructions [Coq90]. No existing constructive foundation has yet superseded the others as the standard one, as Zermelo-Fraenkel axiomatic set theory did for classical mathematics.

Even more such approaches or foundations are not all mutually compatible from a constructive point of view. In order to provide a common basis among this variety of approaches, in joint work with G. Sambin (started from [MS05]) we embarked in the project of building a minimalist foundation to formalize constructive mathematics in a way compatible with the most relevant constructive foundations. A two-level formal system called *minimalist foundation* was completed in [Mai09]. The minimalist foundation is equipped with an extensional level acting as a set theory where to formalize mathematical theorems and proofs, and an intensional one acting as a programming language suitable for the extraction of programs (or constructive contents) from proofs at the extensional level.

In this talk we argue why, when adopting a minimalist approach to constructive mathematics as that formalized in our foundation, the choice for a pointfree approach to topology is not just a matter of convenience or mathematical elegance, but becomes compulsory.

The main reason is that in our minimalist foundation real numbers, either as Dedekind cuts or as Cauchy sequences, do not form a set (and the two representations are distinct due to the absence of choice principles). This result is obtained via a realizability model presented in [Mai12]. This is not surprising since our foundation is compatible even with classical predicative systems.

[1] Based on joint work with G.Sambin.

The only way we see to represent real numbers, both as Dedekind cuts or as Cauchy sequences, in our foundation is to adopt the point-free approach to constructive topology offered by P. Martin-Löf and G. Sambin with the theory of formal topology introduced in [Sam87].

When based on our foundation, the formal topology representations of real numbers allow to distinguish their finitary aspects from the ideal ones as advocated in [Sam12].

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Intuitionistic probability logics

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In late 80's a new type of probability logics, inspired by research in Artificial Inteligence, was developed. Typically, these logics combine propositional logic with modalty-like probabilistic operators togather with Kripke-style semantics. We present an intuitionistic variant, developed in early 2000's, which combines the Intuitionistic propositional logic with probability operators which behave classically. Possible addition of constructive probability will also be disscussed, as well as related work.

On the development of non-classical mathematics

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It is well known that classical theorems, when viewed from a constructive perspective, come apart at the seams. Typically there are crucial assumptions, implicitly made within the classical framework (a result of the classical validity of omniscience principles), which separate out different versions of the same theorem. Brouwerian examples demonstrate where, and why, these theorems are different and the program of constructive reverse mathematics is often able to pinpoint exactly the strength of non-constructive principles necessary for the proving of these theorems. A good example is the intermediate value theorem (IVT)—constructively equivalent to the lesser limited omniscience principle (LLPO). The IVT becomes constructively provable if either stronger hypotheses are introduced, or the conclusion is weakened.

This talk begins to shed some light on what happens in non-classical mathematics more generally (e.g. relevant mathematics, paraconsistent mathematics). In particular, we investigate results concerning order and locatedness—a constructive concept—within a framework of analysis founded on a variety of paraconsistent logic. Again, we find (perhaps unsurprisingly) that one classical theorem has many paraconsistently distinguishable versions. But we find (perhaps surprisingly) that the constructive techniques which play a central role in highlighting these differences can often be adapted to paraconsistency.

Making a detour via intuitionistic theories –Embedding set theories into systems of explcit mathematics–

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The system T_0 of explicit mathematics was suggested by Feferman [1]. While it was first introduced as a system based on intuitionistic logic, later it has been mainly studied as the one based on classical logic. On proof theoretic strengths of it and its extensions, several researchers have been working by comparing these systems with systems of KP and its extensions (cf. Jaeger and Studer
citeJaegerStuder and Jaeger and Strahm [3]). In this talk, we explain the method, suggested by Sato [4], of embedding extensions of (classical) KP into (classical) T_0 and extensions via several intuitionistic set theories, using realizability interpretation, forcing, etc. and consider its application for the embedding between stronger systems. The emphasis should be on the utility of intuitionistic systems for constructing interpretations between classical ones.

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Bishop-style constructive mathematics in type theory — a tutorial

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Errett Bishop developed in his 1967 book *Foundations of Constructive Analysis* (FCA) a powerful informal style of doing mathematics constructively. He later worked on formal systems intended to serve as logical foundations for this mathematical development, but it is probably fair to say that these systems were not entirely successful. The constructive set theory CZF of Aczel and Myhill emerged in the 1970s as a full-fledged system for doing Bishop-style mathematics, and has since been a useful and accepted foundation. These use the standard language of set theory, and are based on (first-order) intuitionistic logic. However, reading the chapter "Set Theory" of FCA, it is clear that Bishop had a more type-theoretic view of the foundations, for instance stating that every set (or type) comes supplied with an explicit notion of equality. Another system emerging in the 1970s — Martin-Löf type theory — was precisely intended serve as such a foundation. Understanding Bishop's sets as types with equivalence relations (so called *setoids*), and functions as operations respecting these equivalence relations, one obtains a faithful and direct development of his set theory. Martin-Löf type theory has become a standard basis for proof assistants such as Coq and Agda, so we can, via Bishop's set theory, formalize constructive mathematics in such systems, and may immediately extract algorithms from proofs.

Contents of lectures:

Part 1: Introduction to type theory: judgement forms, type constructions, interpretation of logic. Bishop's set theory as based on setoids. Choice principles. Examples of formalizations in Coq.

Part 2: Examples from the foundations of analysis. Setoids and dependent families. Formalizing category theory and topology. Further examples in Coq.

Constructive combinatorics of Dickson's lemma

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In the *Combinatorics of Well-Quasi-Orderings* belong theorems stating that a certain quasi-ordering $\preccurlyeq$ on some set W is a well-quasi-ordering. If W is the set of the finite sequences of naturals of arbitrary fixed length and $\preccurlyeq$ is the corresponding pointwise ordering, then one gets the finite versions of Dickson's lemma (DL). This is the first result in a chain of theorems like Higman's lemma, Kruskal's theorem and the Graph Minor Theorem, which refer to appropriately quasi-ordered sets of words, trees and graphs, respectively.

The constructive study of combinatorics of well-quasi-orderings started with [3] and especially with [7] and [14]. Since then many different constructive approaches towards theorems like DL appeared (see e.g., [2], [12], [10], [1]). Here we present some results on the finite and the infinite versions of DL as a case study of constructive combinatorics of well-quasi-orderings within the formal system BIM.

The formal system of *Basic Intuitionistic Mathematics* BIM, introduced in [15], is based on the system of elementary intuitionistic analysis H of Howard and Kreisel [5]. Here we consider a slight variation of BIM [9], which is a bit closer to H. The system BIM is a minimal formal theory of numbers and number-theoretic sequences like Kleene's system M, studied in [8], and the system EL of elementary analysis (see [13]). The language of BIM is a two-sorted language of numerical variables and number-theoretic functions, its logic is two-sorted intuitionistic predicate logic, and its axioms include primitive recursion, full induction and decidable countable choice. Hence one can view BIM as a formalization of a proper part of Bishop's (informal) constructive mathematics BISH [4]. Actually, all proofs within BIM can be read as proofs within BISH. In addition, all proofs within BIM are translated into proofs within TCF (see [11]), therefore they are susceptible to implementation into the proof assistant Minlog [6].

If we present equivalently the finite versions of DL by the formulas $\mathrm{DL}(k, l)$, according to which, for each $k \geq 1$ sequences $\alpha_1, \ldots, \alpha_k$ of naturals there exist $l \geq 2$ indices $i_1 < \ldots < i_l$ on which all α_j's weakly increase, and if $\mathrm{DL}(1, \infty)$ asserts that each sequence has a weakly increasing subsequence (something which is trivially equivalent to $\mathrm{DL}(k, \infty)$), then within

BIM we prove:

(a) $\forall_{k\geq 1}(\forall_{l\geq 2}(\mathrm{DL}(k,l)))$, by induction on k.
(b) $\mathrm{DL}(1,\infty) \leftrightarrow$ LPO, where LPO is the limited principle of omniscience.

The novel feature of the proof of (a) is that it provides a method to determine a bound for each case $\mathrm{DL}(k,l)$, an issue addressed so far only in [1] for the case $\mathrm{DL}(2,2)$. Here we describe the bounds $M(2,2)$ and $M(2,3)$ corresponding to our proofs of $\mathrm{DL}(2,2)$ and $\mathrm{DL}(2,3)$, respectively. In addition to (b) we report on the equivalences between LPO and the infinite pigeonhole principle or the infinite Ramsey theorem. There are also connections between the finite versions of these theorems and the finite versions of DL.

The search for an optimal bound for $\mathrm{DL}(2,2)$, the relation of the finite versions of DL to Higman's lemma on words over an alphabet of two letters, the extension of our methods in corresponding theorems about sequences of ordinals and implementation issues of our results in Minlog are possible topics of further discussion.

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A new approach to constructive pointfree topology: positive topology

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A positive topology is, roughly speaking, a formal topology with a positivity relation rather than a positivity predicate. Positive topologies generalize substantially both the notion of a locale and of a formal topology. One can prove that concrete spaces (that is, topological spaces with continuous relations rather than continuous functions as morphisms, and with a suitable notion of equality) can be embedded into positive topologies. It is not known whether this is possible with the usual formulation of formal topology or with locale theory. If time permits, I will also mention other benefits of the introduction of the positivity relation and of positive topology in general.

Constructive analysis is nonstandard analysis without the middleman

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Nonstandard Analysis (NSA) is often seen as highly "non-constructive in nature". The fundamental axioms of NSA (e.g. Transfer and Standard Part) are indeed non-constructive principles, while the occurrence of ideal objects (e.g. infinitesimals) also seems incompatible with the spirit of constructive mathematics.

These observations notwithstanding, the *practice* of NSA is often very constructive and during my talk, I explore the intimate and concrete connection between NSA and constructive mathematics. As we will see, the BHK-interpretation is the key to removing the "middle man" (i.e. strong axioms such as Transfer and Standard Part) from Nonstandard Analysis to obtain Constructive Analysis.

Proofs, computations and analysis

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Algorithms are viewed as one aspect of proofs in (constructive) analysis. Data for such algorithms are finite or infinite lists of signed digits -1, 0, 1 (i.e., reals as streams), or possibly non well-founded labelled (by lists of signed digits -1, 0, 1) ternary trees (representing uniformly continuous functions). A theory of computable functionals (TCF) suitable for this setting is described. The main tools are (i) a distiction between computationally relevant and irrelevant logical connectives and (ii) simultaneous inductively/coinductively defined predicates. A realizability interpretation of proofs in TCF can be given, and a soundness theorem holds.

Ideal objects for finite methods[1]

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Somewhat miraculously, transfinite methods do work for proving quite a few theorems of a fairly finite nature. The corresponding ideal objects typically turn up towards a contradiction; and therefore, as Hendtlass has observed, do hardly exist. We aim at exploring, explaining, and exploiting this phenomenon: by reducing transfinite proof methods to finite ones, and thus exhibiting the computational content of the virtually inexistent ideal objects. This is work in progress following a bottom-up strategy, i.e. from concrete examples taken from mathematical practice to a possible metamathematics. First case studies have proved successful in the ideal theory of commutative rings [13] and more specifically Banach algebras [5].

More precisely, several theorems that admit short and elegant proofs by contradiction but with Zorn's Lemma have turned out to follow in a direct way from Raoult's Open Induction [9], i.e. transfinite induction limited to Scott–open predicates. Albeit this is yet another classical equivalent of the Axiom of Choice, its use makes possible to eliminate the extremal elements characteristic of any invocation of Zorn's Lemma, and to pass from classical to intuitionistic logic. If moreover the theorem has input data of a sufficiently finite character, then a finite partial order carries the required instance of induction; whence we can get by with at most mathematical induction—and do without any transfinite method whatsoever.

Inspirations come from: the partial realisation in algebra, started by Coquand and Lombardi [3, 6], of the revised Hilbert Programme à la Kreisel and Feferman; the work on infinite combinatorics by U. Berger and Coquand [1, 2]; Coste, Lombardi, and Roy's dynamical algebra [4]; and the interplay between pointwise and point-free topology in Sambin's basic picture [11]. There further are parallels to Maietti and Sambin's two-level foundations with a forget-restore option [8, 7], and to the latter's appeal for allowing the use of ideal objects in real mathematics once this is approved by, say, appropriate conservativity [10, 12].

[1]This partially is joint work with F. Ciraulo, N. Gambino, M. Hendtlass, and D. Rinaldi.

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CONTRIBUTED TALKS

Anti-Specker properties in constructive reverse mathematics

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Specker's theorem, a cornerstone of Russian recursive mathematics, asserts the existence of sequences in the unit interval, called *Specker sequences*, which are eventually bounded away from each point of that interval. Several antitheses of this result — the so-called *anti-Specker properties* — play an important part in constructive reverse mathematics (CRM). These are "semi-constructive" in the sense that they are intuitionistically valid (though not provable in the more minimal system of Bishop-style constructive mathematics, **BISH**), and significant in that they may sometimes be used in the place of sequential compactness to attain a semi-constructive variation upon a classical result.

I examine the role of these properties in the CRM programme, exploring the relationships that they stand in to other principles of real and complex analysis, and the manner in which they may be used in proofs.

A direct proof of open induction on Cantor space using constructive delimited control operators

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Recently, in the context of Constructive Reverse Mathematics, Wim Veldman has shown that, in presence of Markov's Principle (MP), the Open Induction principle on Cantor space (OI) is equivalent to the Double Negation Shift principle (DNS).

Axiom (OI)**.** *Let* α, β *denote infinite binary sequences and* U *an open subspace of Cantor space. Let* $<$ *denote the lexicographic ordering on infinite binary sequences.* Open Induction on Cantor space *is the following principle:*

$$\forall \alpha \, (\forall \beta < \alpha \, (\beta \in U) \to \alpha \in U) \to \forall \alpha (\alpha \in U)$$

Axiom (DNS)**.** *The* Double Negation Shift *is the following principle,*

$$\forall n \neg\neg A(n) \to \neg\neg \forall n A(n) \quad \textit{(for any formula } A(n)\textit{)},$$

where $n \in \mathbb{N}$ *and* $A(n)$ *is any formula.*

Axiom (MP)**.** Markov's Principle *is the statement,*

$$\neg\neg\exists n A_0(n) \to \exists n A_0(n),$$

where $n \in \mathbb{N}$ *and* $A_0(n)$ *is an atomic (decidable) formula.*

In the view of recent works on constructive interpretations of MP and DNS using delimited control operators [2, 3], we reconstruct a direct proof of Veldman's result that derives OI by control operators in the system from [3]. We also point out that any proof of OI from MP and DNS has to use a countable comprehension axiom (axiom of unique countable choice).

In this talk, we will present this work and discuss the computational behavior of the realizer obtained, which is of a different nature than the previous realizer of OI by Ulrich Berger [1].

More information about this work can be found in the manuscript [4].

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Constructive approach to relevant and affine term calculi[1]

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In this paper, we propose a new way to obtain a computational interpretation of some substructural logics, starting from an intuitionistic (i.e. constructive) term calculi with explicit control of resources.

Substructural logics [1] are a wide family of logics obtained by restricting or rejecting some of Gentzen's structural rules, such as weakening, contraction and exchange. The most well known substructural logic is the linear logic of Girard [3], in which, due to the absence of contraction and weakening, each formula appears exactly once in the theorem. The other well known substructural logics are the relevant logic (the one without weakening), the affine logic (without contraction) and the Lambek calculus (without all three structural rules).

From the computational point of view, structural rules of weakening and contraction are closely related to to the control of available resources (i.e. term variables). More precisely, contraction corresponds to the duplication of the variable that is supposed to be used twice in a term, while weakening corresponds to the erasure of an useless variable. These concepts were implemented into several extensions of the lambda calculus [4,5,6,2].

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Here, we use the resource control lambda calculus $\lambda_{\circledR}$, proposed in [2], as a starting point for obtaining computational interpretations of implicative fragments of some substructural logics, namely relevant and affine logic. The corresponding formal calculi are obtained by syntactic restrictions, along with modifications of the reduction rules and the type assignment system. The proposed approach is simpler than the one obtained via linear logic.

The *pre-terms* of $\lambda_{\circledR}$ are given by the following abstract syntax:

$$\text{Pre-terms} \qquad f \;\; ::= \;\; x \,|\, \lambda x.f \,|\, f f \,|\, x \odot f \,|\, x <^{x_1}_{x_2} f$$

where x ranges over a denumerable set of term variables, $\lambda x.f$ is an *abstraction*, ff is an *application*, $x \odot f$ is a *weakening* and $x <^{x_1}_{x_2} f$ is a *contraction*. $\lambda_{\circledR}$-terms are derived from the set of pre-terms by inference rules, that informally specify that bound variables must actually appear in a term and that each variable occurs at most once. Operational semantics of $\lambda_{\circledR}$-calculus is defined by four groups of reduction rules and some equivalencies. The main computational step is the standard β reduction, executed by substitution defined as meta-operator. The group of (γ) reductions performs propagation of contraction into the term. Similarly, (ω) reductions extract weakening out of the terms. This discipline allows us to optimize the computation by delaying duplication of variables on the one hand, and by performing erasure of variables as soon as possible on the other. Finally, the rules in $(\gamma\omega)$ group explain the interaction between explicit resource operators that are of different nature.

The simple types are introduced to the $\lambda_{\circledR}$-calculus in the following figure.

$$\frac{}{x : A \vdash x : A} \; (Ax)$$

$$\frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x.M : \alpha \to \beta} \; (\to_I) \qquad \frac{\Gamma \vdash M : \alpha \to \beta \quad \Delta \vdash N : \beta}{\Gamma, \Delta \vdash MN : \beta} \; (\to_E)$$

$$\frac{\Gamma, x : \alpha, y : \alpha \vdash M : \beta}{\Gamma, z : \alpha \vdash z <^{x}_{y} M : \beta} \; (Cont) \qquad \frac{\Gamma \vdash M : \beta}{\Gamma, x : \alpha \vdash x \odot M : \beta} \; (Weak)$$

In the obtained system $\lambda_{\circledR} \to$, weakening is explicitly control by the choice of the axiom, while the control of the contraction is managed by implementing context-splitting style (i.e. requiring that Γ, Δ represents disjoint union of the two bases).

Modifications of the $\lambda_{®} \rightarrow$ system can provide the computational interpretation of some substructural logics, different from the usual approach via linear logic. For instance, if one excludes the $(Weak)$ rule but preserves the axiom that controls the introduction of variables, the resulting system would correspond to the logic without weakening and with explicit control of contraction i.e. to the variant of implicative fragment of relevance logic. Similarly, if one excludes the $(Cont)$ rule, but preserves context-splitting style of the rest of the system, correspondence with the variant of the logic without contraction and with explicit control of weakening i.e. implicative fragment of affine logic is obtained. Naturally, these modifications also require certain restrictions on the syntactic level, changes in the definition of terms and modifications of operational semantics as well.

We also proposed intersection type assignment systems for both the $\lambda_{®}$-calculus and its substructural restrictions, and proved that those systems completely characterize strongly normalising terms of all three calculi.

Although the proposed systems may be considered naive due to the fact that they only correspond to implicative fragments of relevant and affine logics and therefore are not able to treat characteristic split conjunction and disjunction connectives, they could be useful as a simple and neat logical foundation for the programming specific relevant and affine programming languages.

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Functorial embedding of compact uniform spaces into formal topologies

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Our work is an attempt to bridge the gap between the two notions of compactness in constructive topologies, that of a Bishop metric space[2] which is defined to be compact if it is complete and totally bounded, and that of a formal topology[6] where compactness is defined in the sense of Heine-Borel. It is well-know that the point-wise topology associated with a Bishop compact metric space cannot be Heine-Borel compact; if it were, it would imply Fan theorem, which is not acceptable constructively. Putting it another way, the formal topology associated with a Bishop compact metric space via a well-known adjunction between the category of point-set topological spaces and that of formal topologies cannot be compact as a formal topology. Thus, a question arises as to how the two notions of compactness are related.

Inspired by the pioneering work of Erik Palmgren[5], in which he constructed an embedding from the category of Bishop locally compact metric spaces into that of locally compact regular formal topologies, we have constructed an embedding from the category of Bishop compact uniform spaces and uniformly continuous functions into that of compact 2-regular[1] formal topologies. Restricted to the full subcategory of compact metric spaces, this gives another embedding from the category of Bishop compact metric spaces into the category of formal topologies. However, a connection between our work and that of Palmgren's is still unclear.

We build on the work of Christopher Fox[3], where he introduced a constructive version of uniform formal topology (formal topology with a compatible covering uniformity), defined a point-free completion functor from the category of uniform formal topologies to that of complete uniform formal topologies, and showed that a uniform formal topology is totally bounded if and only if its point-free completion is compact.

We defined a slightly simplified notion of uniform formal topology based on Fox's and showed the similar results as above. Moreover, we noted that

[1]The notion of 2-regularity is weaker than the usual notion of regularity of formal topologies.

by post composing the completion functor with the functor from the category of uniform spaces to that of uniform formal topologies (which is similar to the familiar functor from the category of topological spaces to that of formal topologies), we can obtain a full and faithful functor from the category of Bishop compact uniform spaces to that of compact uniform formal topologies. Furthermore, we showed that any formal topology map between uniform formal topologies with compact domain is uniformly continuous with respect to the point-free uniform structures. Since every uniform formal topology is 2-regular, the composition thus defines an embedding from the category of compact uniform spaces to that of compact 2-regular formal topologies.

In particular, a certain compact formal topology arises as a point-free completion of the point-wise topology associated with a Bishop compact uniform space. For example, the embedding sends the (point-set) Cantor space to the formal Cantor space, and there is a bijective correspondence between the uniformly continuous functions between Cantor spaces and formal topology maps between formal Cantor spaces.

Our work is constructive (intuitionistic and predicative) and choice free, so we believe that it can be formalized in constructive frameworks such as Martin-Löf's type theory[4] and Aczel's constructive set theory CZF[1].

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About decidability of a problem in the theory of analytic inequalities

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In the presentation it is considered the set S, of real functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ built up from integers and x using addition, multiplication and composing with sin function. Then it is impossible to decide, given $f \in S$, whether f is everywhere nonnegative. It stands out subclass S_0 for which it is possible to give a general procedure of the previous decidability problems. Results are connected with some current results from the theory of analytic inequalities.

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Types for communication behaviour

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We present novel type theories, built on the concept "types-as-processes", that have arisen from process calculi, influenced by communication-centered software development.

In 1980, Milner [7] introduced a process calculus, Calculus of Communicating System (CCS), as a model of concurrent computation and interaction.

In 1992, Milner, Parrow and Walker [8] developed the Pi Calculus, following work of Engberg and Nielsen, extending the CCS in order to be able to express mobility.

Taking Pi Calculus as a core model, last two decades brought a significant number of calculi for modeling specific behaviours, as well as corresponding type systems. These are, for example, dPi Calculus, Ambient Calculus, XdPi Calculus, Spi Calculus etc. In our research, we put emphasis on security types for dynamic web data, with recent application to the concept of Linked Data ([1], [2], [3], [4]).

Finally, we present behavioural type theories as expressive formalisms for structured communication such as commmunication protocols and contracts. In 1993, Honda et al. ([5], [6]) introduced session types to describe the structure of communication. More precisely, session types represent sequences of messages that are to be exchanged on a session private channel. They are expected to be crucial for development of reliable concurrent and distributed systems.

This is result of joint work with Mariangiola Dezani, Silvia Ghilezan, Svetlana Jakšić and Daniele Varacca.

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