Probability logics

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- Group for Probability logics
- Leader: professor Miodrag Rašković
- Members: Zoran Marković, Zoran Ognjanović, ... (≥ 10)
- Project: Representations of logical structures and their application in computer science, 2006-10, Fundamental Research project funded by Serbian Ministry of Sciences and Technological Development (≥ 30 researchers, $\geq 25\%$ of PhD-students)
- http://www.mi.sanu.ac.rs/projects/144013e.htm

Example

Knowledge base:

if A_1 then B_1

if A_2 then B_2

if A_3 then B_3

...

from an expert system (MYCIN, PROSPECTOR, ...)

- Knowledge base with certainty factors:

  if A_1 then B_1 (cf c_1)
  if A_2 then B_2 (cf c_2)
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- Uncertain knowledge: from statistics, our experiences and beliefs, etc.
- Nils Nilsson (Probabilistic logic, AI 28, 1986): generalization of classical logic for dealing with uncertainties
- To check consistency of (finite) sets of sentences.
- To deduce probabilities of conclusions from uncertain premisses.

- How Modus Ponens can be generalized when one assigns probabilities:

$$Prob(A) = a$$
$$Prob(A \rightarrow B) = b$$
$$Prob(B) =?$$

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$$\begin{aligned} &Prob(A) = a \\ &Prob(A \rightarrow B) = b \\ &Prob(B) =? \end{aligned}$$

- $Prob(A) + Prob(A \rightarrow B) - 1 \leq P(B) \leq Prob(A \rightarrow B)$
- Similar to Boole's procedure

Proof-theoretic approaches:

- H. Gaifman. A Theory of Higher Order Probabilities. In: *Proceedings of the Theoretical Aspects of Reasoning about Knowledge* (edts. J.Y. Halpern), Morgan-Kaufmann, San Mateo, California, 275–292. 1986.
- M. Fattorosi-Barnaba and G. Amati. Modal operators with probabilistic interpretations I. *Studia Logica* 46(4), 383–393. 1989.
- R. Fagin, J. Halpern and N. Megiddo. A logic for reasoning about probabilities. *Information and Computation* 87(1-2):78 – 128. 1990.
- M. Rašković. Classical logic with some probability operators. *Publications de l'Institut Mathématique*, n.s. 53(67), 1 – 3. 1993.
- R. Fagin and J. Halpern. Reasoning about knowledge and probability. *Journal of the ACM*, 41(2):340–367, 1994.
- A. Frish and P. Haddawy. Anytime deduction for probabilistic logic. *Artificial Intelligence* 69, 93 – 122. 1994.

- The probabilistic logics allow strict reasoning *about* probabilities using well-defined syntax and semantics
- Formulas in these logics remain either true or false
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- The probability that a particular bird A flies is at least 0.75
- $P_{\geq 0.75} Fly(A)$
- $\mu(\{w : w \models Fly(A)\}) \geq 0.75$

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- Basic probabilistic formula: $P_{\geq s}\alpha$ for $\alpha \in For_C$, $s \in Q \cap [0, 1]$
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- $P_{\geq s}P_{\geq t}\alpha \notin For$
- $\beta \vee P_{\geq s}\alpha \notin For$

- A probabilistic model $M = \langle W, H, \mu, v \rangle$:
  - W is a nonempty set of elements called worlds,
  - H is an algebra of subsets of W,
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- Measurable models
  - $\alpha \in For_C$
  - $[\alpha] = \{w \in W : w \models \alpha\}$
  - $[\alpha] \in H$

Satisfiability:

- if $\alpha \in For_C$, $M \models \alpha$ if $(\forall w \in W)v(w)(\alpha) = \top$
- $M \models P_{\geq s}\alpha$ if $\mu([\alpha]_M) \geq s$,
- if $A \in For_P$, $M \models \neg A$ if $M \not\models A$,
- if $A, B \in For_P$, $M \models A \wedge B$ if $M \models A$ and $M \models B$.

A set of formulas $F = \{A_1, A_2, \ldots\}$, is satisfiable if there is a model M, $M \models A_i$, $i = 1, 2, \ldots$.

- Providing a sound and complete axiom system
  - the simple completeness (every consistent formula is satisfiable, $\models A$ iff $\vdash A$)
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- Compactness (a set of formulas is satisfiable iff every finite subset is satisfiable).

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- M satisfies every F_k, but does not satisfy F
- finitary axiomatization + extended completeness imply compactness
- finitary axiomatization for a real valued probabilistic logic: there are consistent sets that are not satisfiable

Restrictions:

- on ranges of probabilities: $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$
- infinitary axiomatization

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- axioms for probabilistic reasoning
  - $P_{\geq 0}\alpha$
  - $P_{\leq r}\alpha \rightarrow P_{<s}\alpha$, $s > r$
  - $P_{<s}\alpha \rightarrow P_{\leq s}\alpha$
  - $(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}(\neg(\alpha \wedge \beta))) \rightarrow P_{\geq \min(1, r+s)}(\alpha \vee \beta)$
  - $(P_{\leq r}\alpha \wedge P_{<s}\beta) \rightarrow P_{<r+s}(\alpha \vee \beta)$, $r + s \leq 1$

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  - $(P_{\leq r}\alpha \wedge P_{<s}\beta) \rightarrow P_{<r+s}(\alpha \vee \beta)$, $r + s \leq 1$
- inference rules
  - From Φ and $\Phi \rightarrow \Psi$ infer Ψ.
  - From α infer $P_{\geq 1}\alpha$.
  - From $A \rightarrow P_{\geq s - \frac{1}{k}}\alpha$, for every integer $k \geq \frac{1}{s}$, and $s > 0$ infer $A \rightarrow P_{\geq s}\alpha$.

- Proof from the set of formulas ($F \vdash \varphi$):
  - at most denumerable sequence of formulas φ_0, φ_1, ..., φ, such that every φ_i is an axiom or a formula from the set F, or it is derived from the preceding formulas by an inference rule

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- A formula φ is a *theorem* ($\vdash \varphi$) if it is deducible from the empty set.
- A set F of formulas is *consistent* if there are at least a classical formula and at least a probabilistic formula that are not deducible from F.
- Object language is countable, formulas are finite. Only proofs are allowed to be infinite.

The main proof-theoretical results (F - set of formulas):

Theorem (Deduction)

$F, \phi \vdash \psi$ iff $F \vdash \phi \to \psi$.

Theorem (Extended completeness (1))

F is consistent iff F is satisfiable.

Theorem (Extended completeness (2))

$F \vdash \phi$ iff $F \models \phi$.

Transfer of the results to the case of σ-additive probabilities.

Decidability of the satisfiability problem for the probabilistic logic (PSAT):

Theorem

PSAT is NP-complete.

Early papers:

- G. Georgakopoulos, D. Kavvadias, and C. Papadimitriou. Probabilistic satisfiability. *Journal of Complexity* 4(1):1–11. 1988.
- R. Fagin, J. Halpern and N. Megiddo. A logic for reasoning about probabilities. *Information and Computation* 87(1-2):78 – 128. 1990.
- B. Jaumard, P. Hansen, and M. P. de Aragao. Column generation methods for probabilistic logic. *ORSA Journal on Computing* 3:135–147. 1991.

- $\bigwedge_{j=1}^{k} a_1^j Prob(\mathrm{CDNF}(\alpha_1^j)) + \ldots + a_{n_j}^j Prob(\mathrm{CDNF}(\alpha_{n_j}^j)) \ \rho_j \ c^j$
- $\rho_j \in \{\geq, <\}$
- a_i^j's and c^j's are rational numbers
- $\mathrm{CDNF}(\alpha)$ - the complete disjunctive normal form of α

- $Prob(p \rightarrow q) + Prob(p) \geq 1.7 \wedge Prob(q) \geq 0.6$

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  $\wedge$
  $Prob(p \wedge q) + (\neg p \wedge q)) \geq 0.6$
- The formula is satisfiable iff the same holds for the linear system:

$$\mu(p \wedge q) + \mu(p \wedge \neg q) + \mu(\neg p \wedge q) + \mu(\neg p \wedge \neg q) = 1$$

$$\begin{aligned} &\mu(p \wedge q) \geq 0 \\ &\mu(p \wedge \neg q) \geq 0 \\ &\mu(\neg p \wedge q) \geq 0 \\ &\mu(\neg p \wedge \neg q) \geq 0 \end{aligned}$$

$$\begin{aligned} &\mu(p \wedge \neg q) + \mu(\neg p \wedge q) + \mu(\neg p \wedge \neg q) + 2\mu(p \wedge q) \geq 1.7 \\ &\mu(p \wedge q) + \mu(\neg p \wedge q) \geq 0.6. \end{aligned}$$

Genetic algorithms

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- general problem solving methods inspired by processes of natural evolution
- use populations of individuals (possible solutions for the problem)
- define the corresponding evaluation functions assigning fitness values to individuals
- apply the genetic operators (selection, crossover and mutation) to populations to improve the corresponding average fitness values from each generation to subsequent

```
InputData();
PopulationInit();
while ( not FinishedGA() ) {
        for ( i = 0 ; i < N_pop ; i++) p_i = ObjectiveFunction();
        HeuristicImprovement();
        ComputeFitnesses();
        Selection();
        Crossover();
        Mutation();
}
OutputResults();
```

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- an individual is a description of a probabilistic model
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- solve problems with 200 propositional letters (systems with 2^{200} variables, models with 2^{200} states

Extensions:

- logics with richer languages:
  - conditional probability operators $CP_{\geq s}(A, B)$
  - a operator for qualitative probability $A \lhd B$
  - first order probability logics
  - iterations of probabilistic operators

Overview:

- http://cms.uns.ac.rs/deuks/uploads/Outcomes/Probability Logics

Extensions:

- logics with richer languages:
  - conditional probability operators $CP_{\geq s}(A, B)$
  - a operator for qualitative probability $A \lhd B$
  - first order probability logics
  - iterations of probabilistic operators
- different ranges of probabilistic functions:
  - finite ranges $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$
  - infinitesimals: $CP_{\approx 1}(A, B)$, defaults,
  - partially ordered countable commutative monoids,
  - ...

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Thank you for your attention.