

Theoretical Computational Sciences
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The description of a transmission control protocol by q–calculus

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1 Basics of q -calculus

The first letter in the name " q -calculus" can be considered like:

- (1) the first letter of "quantum" ("**quantum calculus**");
- (2) the parameter q which is included in all further discussions;
- (3) the symbol of **q -world**, which always can be returned to the well-known mathematical world when $q \nearrow 1$.

[1] G. ANDREWS, R. ASKEY, AND R. ROY, "*Special functions*", Cambridge University Press, Cambridge, 1999.

[2] G. GASPER, M. RAHMAN, "*Basic Hypergeometric Series*", Encyclopedia of Math. and its Appl. **96**, Cambridge, 2004.

1.1 Our references

- [1] P. RAJKOVIĆ, S.D. MARINKOVIĆ, M.S. STANKOVIĆ, "*Differential and integral calculus of basic hypergeometric functions*", **monograph**, Niš, 2008.
- [2] P.M. RAJKOVIĆ, S.D. MARINKOVIĆ, M.S. STANKOVIĆ, *A generalization of the concept of q -fractional integrals*, Acta Mathematica Sinica, English series **25** No. 10 (2009) 1635–1646.
- [3] W. KOEPF, P.M. RAJKOVIĆ, S.D. MARINKOVIĆ, *Functions satisfying holonomic q -differential equations*, Journal of Difference Equations and Applications, **13** No 7 (2007) 621–638.

- [4] S.D. MARINKOVIĆ, P.M. RAJKOVIĆ, M.S. STANKOVIĆ, *The inequalities for some types q -integrals*, *Computers and Mathematics with Applications* **56** (2008) 2490–2498.
- [5] P.M. RAJKOVIĆ, S.D. MARINKOVIĆ, M.S. STANKOVIĆ, *On q -orthogonal polynomials over a collection of complex origin intervals related to little q -Jacobi polynomials*, *The Ramanujan Journal*, **12** No. 2 (2006) 245–255.
- [6] P.M. RAJKOVIĆ, S.D. MARINKOVIĆ, M.S. STANKOVIĆ, *On q -Newton-Kantorovich method for solving systems of equations*, *Applied Mathematics and Computation*, vol. 168 (2005), 1432-1448.

1.2 The properties of q -numbers

For any complex number λ , its q -analog is

$$[\lambda]_q := \frac{1 - q^\lambda}{1 - q}, \quad \lim_{q \nearrow 1} [\lambda]_q = \lambda.$$

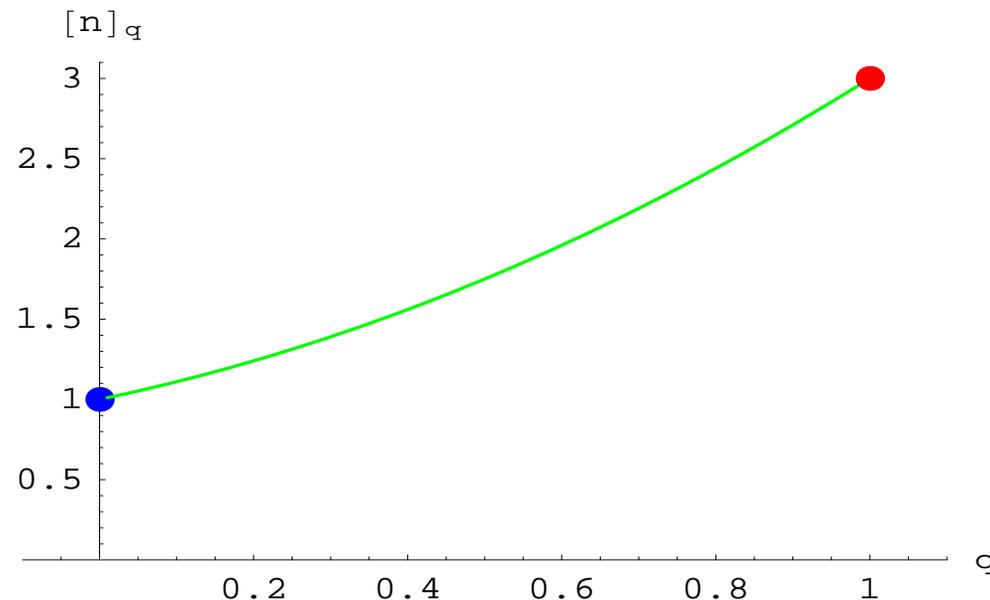


Figure 1. Number $[3]_q = 1 + q + q^2$ when $q \uparrow 1$.

For addition, it is valid

$$[m]_q + q^m [n]_q = [m + n]_q.$$

For a positive integer number $[n]_q$, **q -factorial** is

$$[0]_q! := 1, \quad [n]_q! := [n]_q [n - 1]_q \cdots [1]_q.$$

The **q -binomial coefficients** can be defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n - k]_q!}.$$

It is a polynomial in q of degree $k(n - k)$.

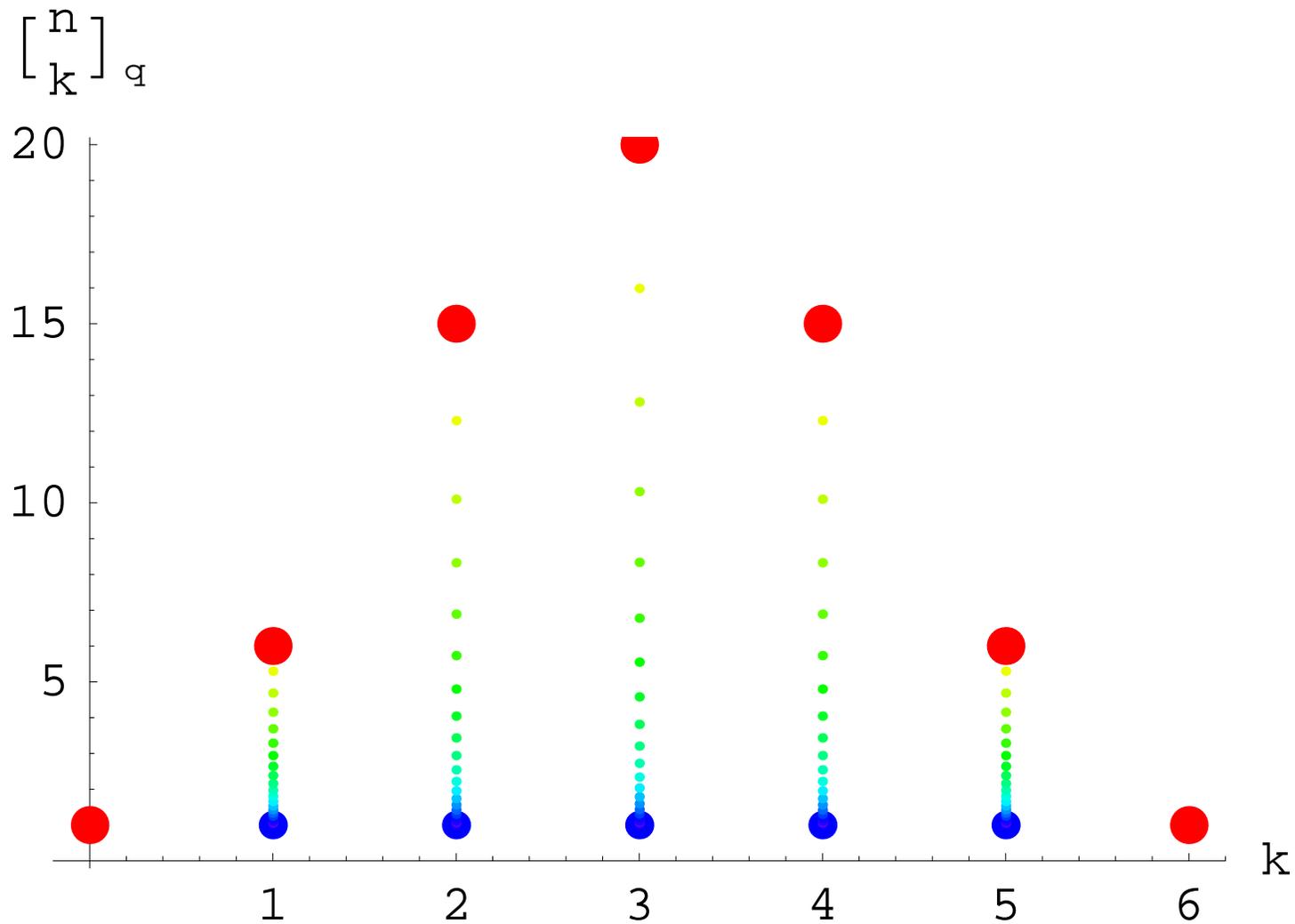


Figure 2. q -Binomial coefficients.

1.3 The Pochhammer symbol

The Pochhammer's symbol is

$$(a)_0 := 1, \quad (a)_k := \prod_{i=1}^k (a + i - 1), \quad (k \in \mathbb{N}).$$

In order to find its q -analog, we consider the product

$$\prod_{i=1}^k [\alpha + i - 1]_q = \prod_{i=1}^k \frac{1 - q^{\alpha+i-1}}{1 - q}.$$

Putting $a = q^\alpha$, we can define

1.4 The q -Pochhammer symbol

the q -Pochhammer symbol of a number a like

$$(a; q)_0 := 1, \quad (a; q)_k := \prod_{i=1}^k (1 - aq^{i-1}), \quad k = 1, 2, \dots$$

In the limit case, we have

$$\lim_{q \nearrow 1} \frac{(q^\alpha; q)_k}{(1 - q)^k} = (\alpha)_k.$$

By the properties

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty},$$

we can introduce

$$(a; q)_\lambda := \frac{(a; q)_\infty}{(aq^\lambda; q)_\infty} \quad (\lambda \in \mathbb{C}),$$

where we take the main branch of the complex function x^λ .

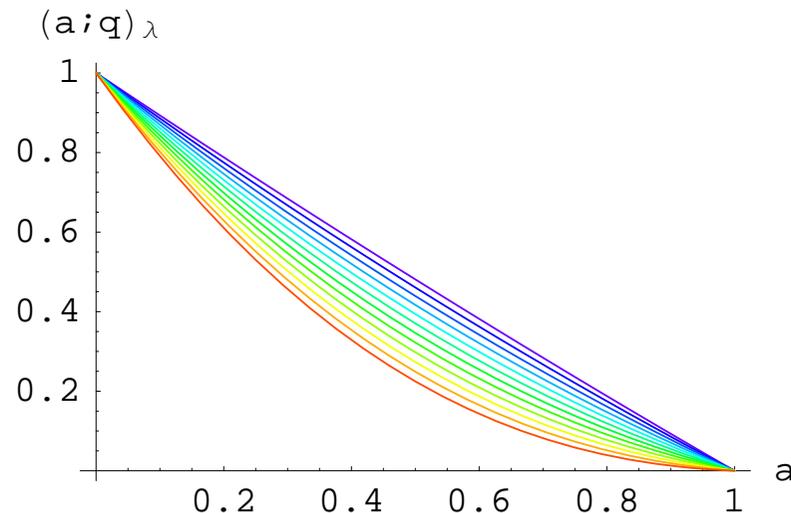


Figure 3. The function $(a; q)_\lambda$ for $\lambda = 2.5$ when $q \uparrow 1$.

1.5 The q -binomial theorem

The binomial formula

$$(x + y)^m = \sum_{n=0}^m \binom{m}{n} x^n y^{m-n}.$$

and its generalization, the binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1 - z)^{-a} \quad (|z| < 1, a \in \mathbb{R}).$$

have their analog in q -calculus.

We will start with

$$f(a, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n \quad (|z| < 1)(|q| < 1)(a, q, z \in \mathbb{C}).$$

Notice

$$f(q, z) = \sum_{n=0}^{\infty} z^n = (1 - z)^{-1}.$$

Since

$$f(a, z) = 1 + \sum_{n=1}^{\infty} \frac{(aq; q)_{n-1}}{(q; q)_n} (1 - a) z^n = (1 - az) f(aq, z)$$

by repeating this step $(n - 1)$ -times, we yield

$$f(a, z) = (1 - az)(1 - aqz) \cdots (1 - aq^{n-1}z) f(aq^n, z).$$

When $n \rightarrow \infty$, we get

$$f(a, z) = (az; q)_{\infty} f(0, z).$$

Putting $a = q$, we have

$$f(0, z) = \frac{f(q, z)}{(qz; q)_{\infty}} = \frac{1}{(1 - z)(qz; q)_{\infty}} = \frac{1}{(z; q)_{\infty}}.$$

Hence

$$f(a, z) = \frac{(az; q)_{\infty}}{(z; q)_{\infty}} .$$

So, we have proved the **q -binomial theorem**

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_{\infty}}{(z; q)_{\infty}} \quad (|z| < 1, |q| < 1).$$

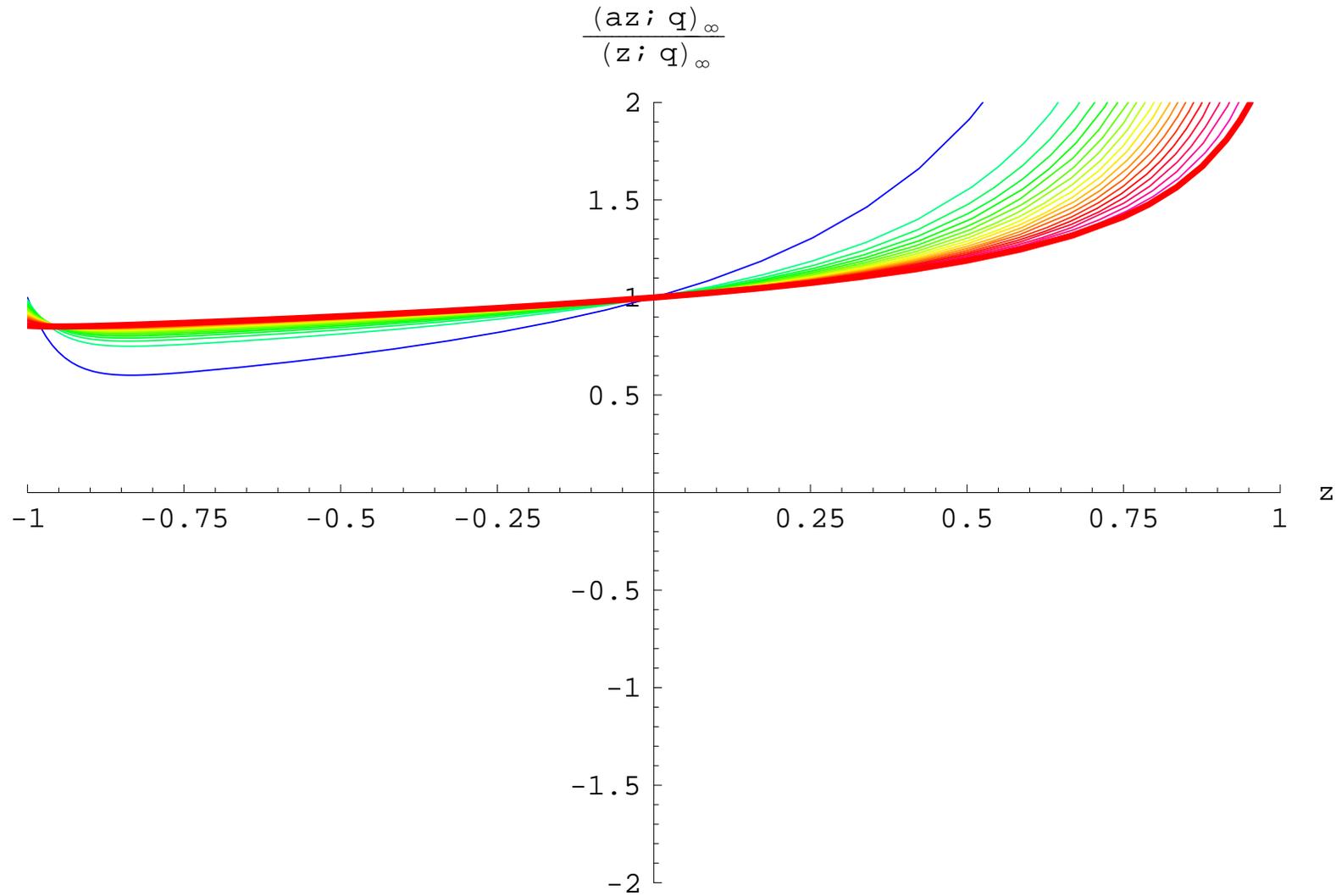


Figure 4. q -Binomial theorem for $a = 1/4$.

1.6 The q -hypergeometric function

The q -hypergeometric function is given by

$$\begin{aligned}
 {}_r\Phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right) \\
 = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k} \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r} \frac{z^k}{(q; q)_k} .
 \end{aligned}$$

It is valid

$$\begin{aligned}
 \lim_{q \nearrow 1} {}_r\Phi_s \left(\begin{matrix} q^{a_1}, \dots, q^{a_r} \\ q^{b_1}, \dots, q^{b_s} \end{matrix} \middle| q; (q-1)^{1+s-r} z \right) \\
 = {}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| z \right) .
 \end{aligned}$$

1.7 The q -special functions

From the q -gamma function $\Gamma_q(x)$, we expect that it has the following properties

$$\Gamma_q(1) = 1, \quad \Gamma_q(x + 1) = [x] \cdot \Gamma_q(x)$$

and

$$\lim_{q \uparrow 1} \Gamma_q(x) = \Gamma(x) = \int_0^{\infty} t^x e^{-t} dt .$$

That is why it is defined by

$$\Gamma_q(x) := (q; q)_{x-1} \cdot (1 - q)^{1-x} .$$

1.8 The q -exponential function

We can introduce the analogs of the elementary mathematical functions in the following way:

the **small and the big q -exponential function**

$$e_q(z) := {}_1\Phi_0\left(\begin{matrix} 0 \\ - \end{matrix} \middle| q; z\right) = \sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k},$$

$$E_q(z) := {}_0\Phi_0\left(\begin{matrix} - \\ - \end{matrix} \middle| q; -z\right) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{(q; q)_k} z^k.$$

It is valid

$$e_q(z) \cdot E_q(-z) = 1,$$

$$\lim_{q \nearrow 1} e_q((1-q)z) = \lim_{q \nearrow 1} E_q((1-q)z) = e^z.$$

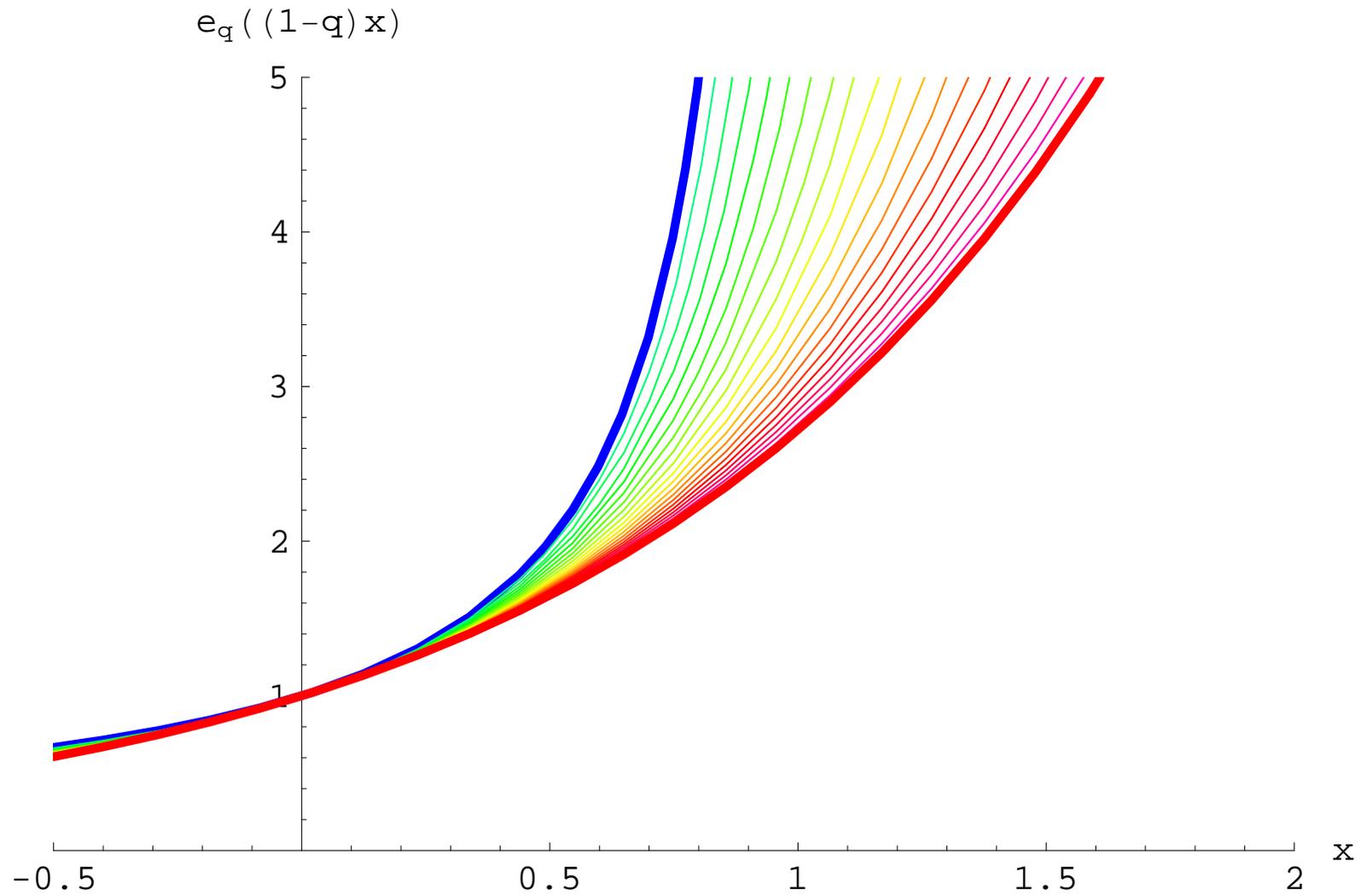


Figure 5.

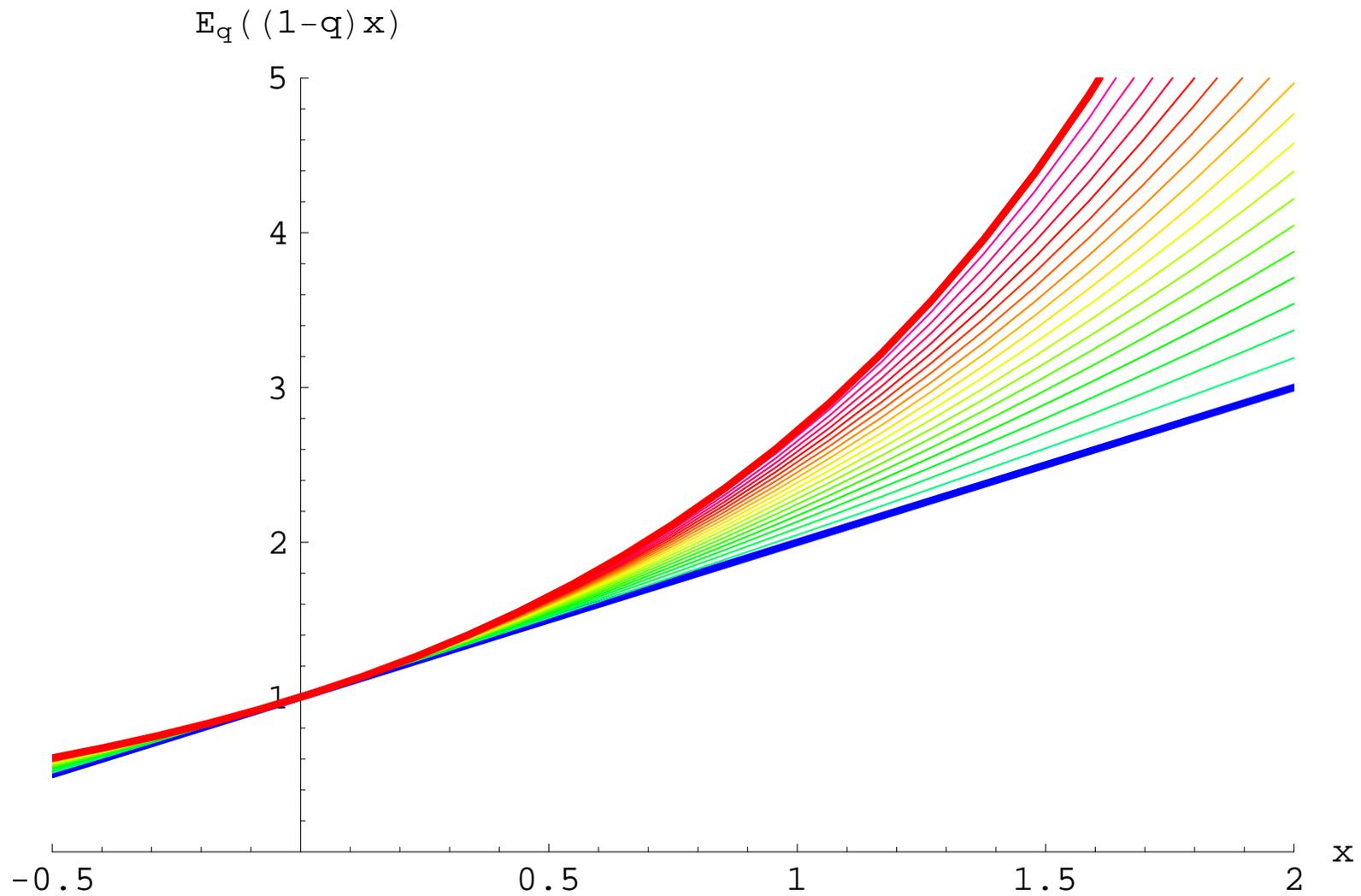


Figure 6.

the small q -trigonometric functions

$$\sin_q(z) := \frac{e_q(iz) - e_q(-iz)}{2i} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(q; q)_{2k+1}} z^{2k+1},$$

$$\cos_q(z) := \frac{e_q(iz) + e_q(-iz)}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(q; q)_{2k}} z^{2k}.$$

Similarly, the **big q -trigonometric functions** are

$$\text{Sin}_q(z) := \frac{E_q(iz) - E_q(-iz)}{2i},$$

$$\text{Cos}_q(z) := \frac{E_q(iz) + E_q(-iz)}{2}.$$

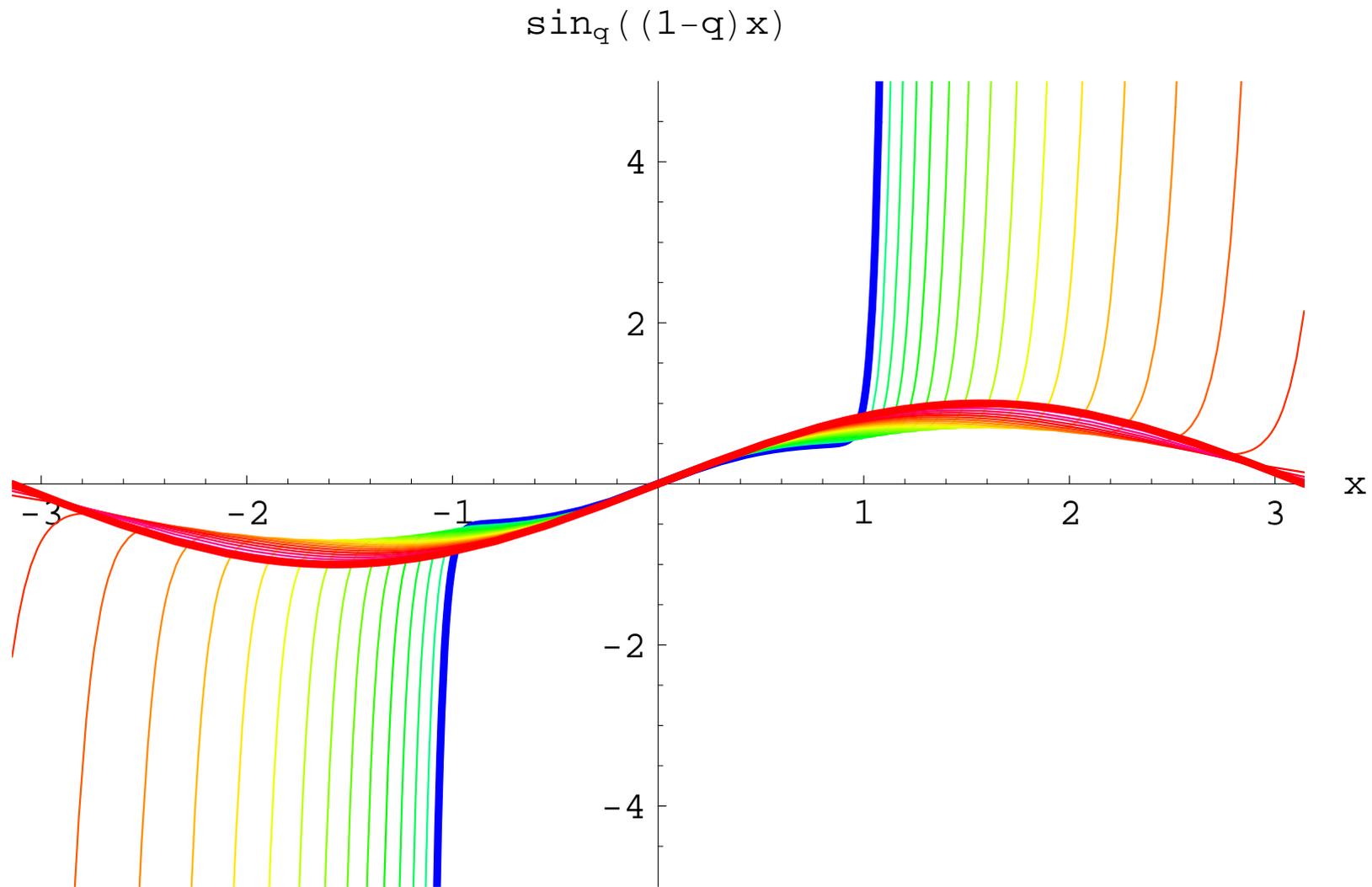


Figure 7.

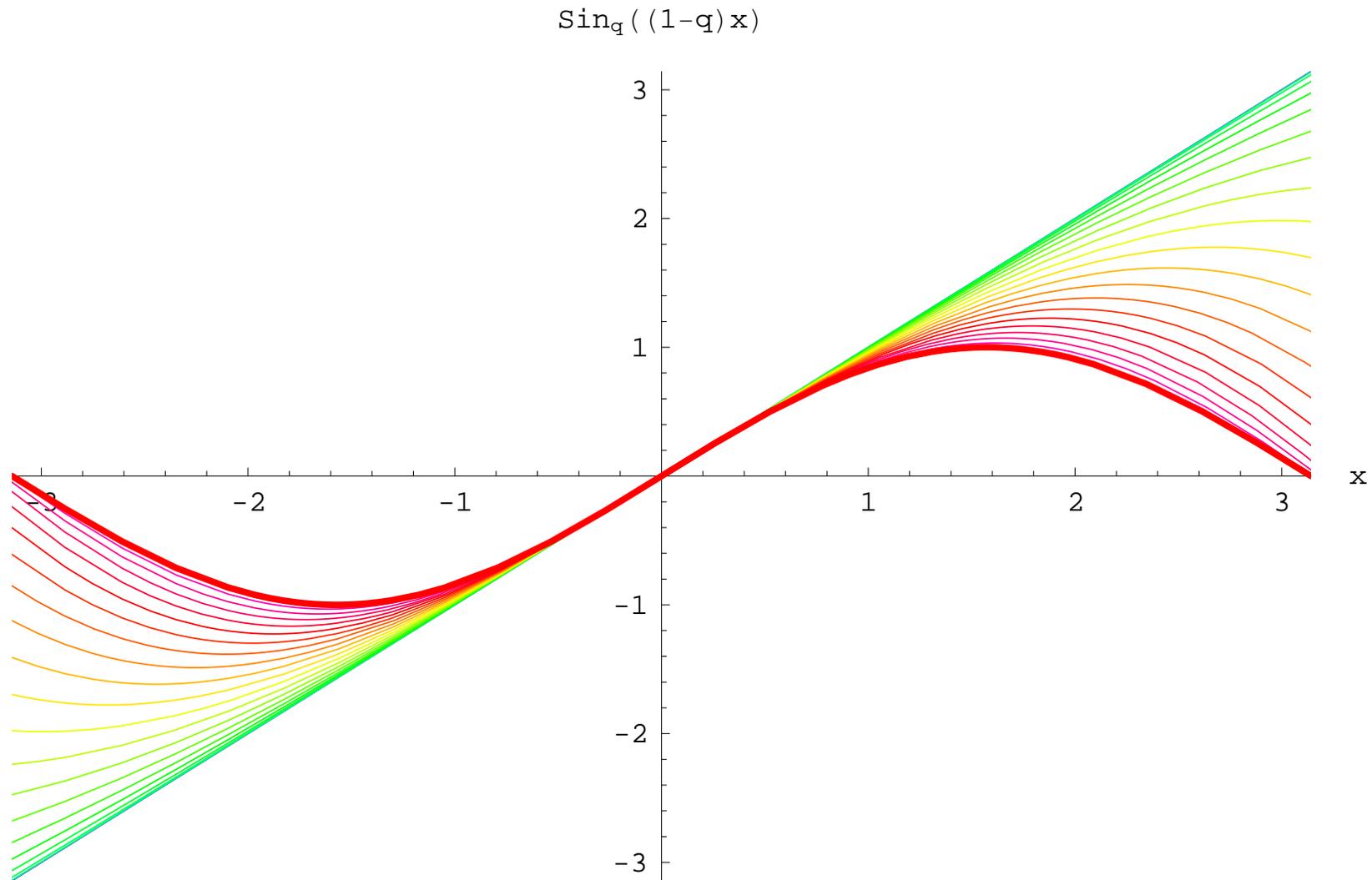


Figure 8.

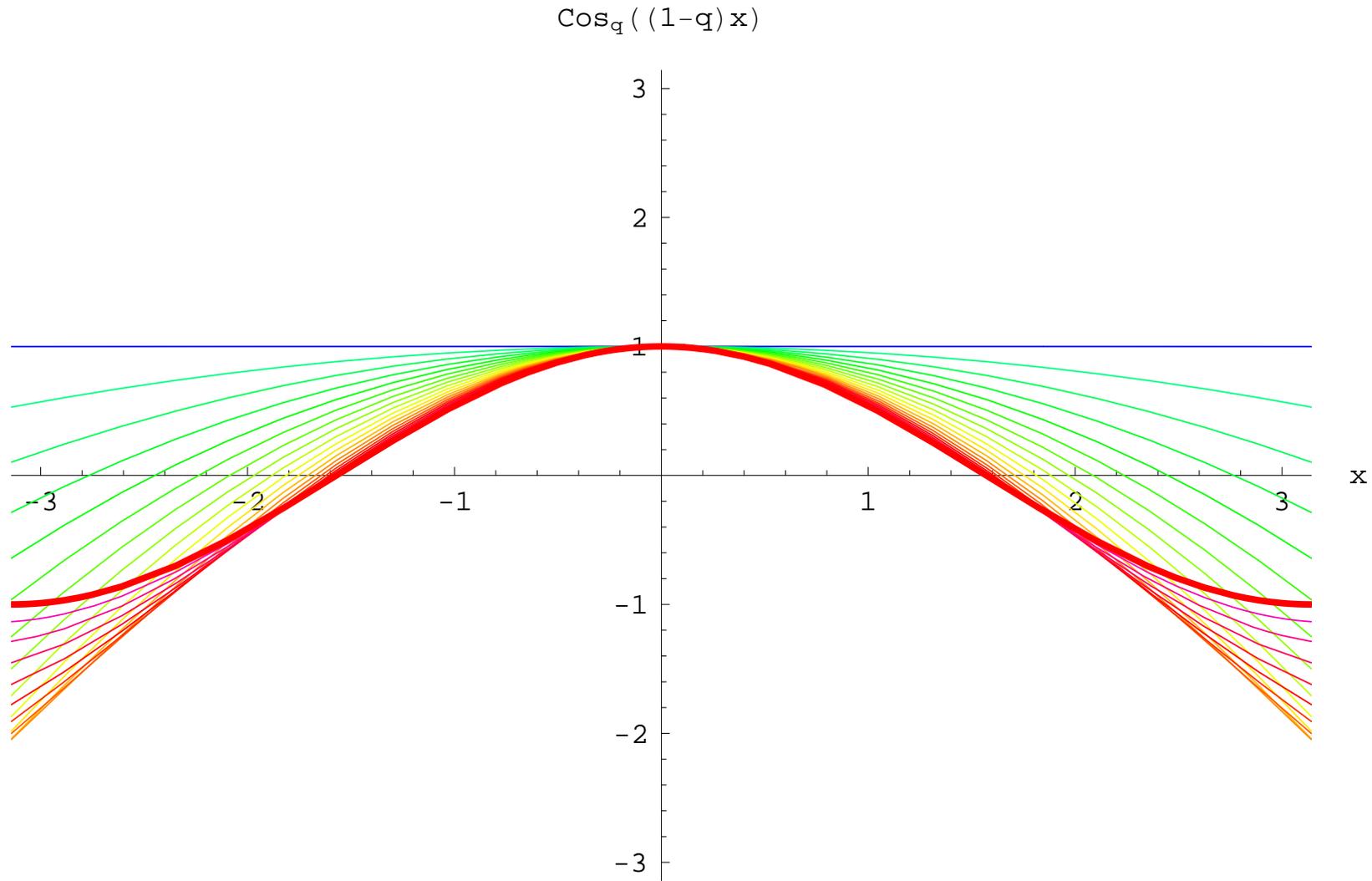


Figure 9

2 Basics of stochastic processes

2.1 Probability space

A probability space associated with a random experiment is a triple (Ω, \mathcal{F}, P) , where:

- Ω is the set of all possible outcomes of the random experiment, and it is called the sample space;
- \mathcal{F} is a family of subsets of Ω which has the structure of a σ -field;
- $P : \mathcal{F} \rightarrow [0, 1]$ is a function which associates a number $P(A)$ to each set $A \in \mathcal{F}$ and it is called a probability measure.

D. NUALART, "Seminar on Stochastic Analysis, Random Fields and Applications", Ascona 2002.

2.2 Random variable

A random variable is a mapping $X : \Omega \rightarrow \mathbb{R}$ which is \mathcal{F} -measurable. The mathematical expectation of a random variable X is defined as the integral of X with respect to the probability measure P :

$$E(X) = \int_{\Omega} X \, dP .$$

The function defined by

$$F_X(x) = P(X \leq x) : \mathbb{R} \rightarrow [0, 1]$$

is called the distribution function of the random variable X .

2.3 The discrete random variable

We say that a random variable X is discrete if it takes a finite or countable number of different values x_k with probability p_k , i.e.

$$P(X = x_k) = p_k \quad (k \in \mathbb{N}_0).$$

Its expectation will be

$$E(X) = x_1 p_1 + x_2 p_2 + \cdots .$$

The probability-generating function of a discrete random variable is a power series representation of the probability mass function of the random variable, i.e.,

$$G(z) = E(z^X) = \sum_{k=0}^{\infty} p_k z^k .$$

2.4 The Poisson distribution

The Poisson distribution is a discrete probability distribution that expresses the probability of a number of events occurring in a fixed period of time if these events occur with a known average rate and independently of the time since the last event.

The certain random variables X count the number of discrete occurrences that take place during a time-interval of given length. If the expected number of occurrences in this interval is λ , then the probability that there are exactly k is equal to

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad (\lambda > 0, k \in \mathbb{N}_0) .$$

The Poisson distribution can be applied to systems with a large number of possible events, each of which is rare.

2.5 The absolutely continuous random variable

We will say that a random variable X has a probability density $f_X(t)$ if $f_X : \mathbb{R} \rightarrow \mathbb{R}^+$ and

$$P(a < X < b) = \int_a^b f_X(t) dt \quad (\forall a, b : a < b) .$$

Random variables admitting a probability density are called absolutely continuous.

Its distribution function is

$$F_X(t) = \int_{-\infty}^t f_X(y) dy .$$

The Laplace transform of the random variable X with density function $f_X(t)$ is

$$L[X(t)] = \int_0^{\infty} f_X(t) e^{-st} dt .$$

2.6 The random variable with exponential distribution

A random variable $X : \Omega \rightarrow (0, \infty)$ has exponential distribution of parameter λ if

$$P(X > t) = e^{-\lambda t} \quad (\forall t > 0) .$$

Then X has a density function

$$f_X(t) = \lambda e^{-\lambda t} \mathbf{1}_{(0, \infty)}(t) .$$

The mean of X is given by

$$E(X) = \frac{1}{\lambda} .$$

Its Laplace transform is

$$L[X(t)] = \int_0^{\infty} \lambda e^{-\lambda t} e^{-st} dt = \frac{\lambda}{\lambda + s} .$$

A random variable $X : \Omega \rightarrow (0, \infty)$ has an exponential distribution if and only if it has the following memoryless property

$$P(X > s + t \mid X > s) = P(X > t) \quad (\forall s \geq 0, \forall t \geq 0) .$$

2.7 The Stochastic Process

A stochastic process with state space S is a collection of random variables $(X_t, t \in T)$ defined on the same probability space (Ω, \mathcal{F}, P) .

The set T is called its parameter set.

If $T = \mathbb{N}_0$, the process is said to be a discrete parameter process.

In the continuous case, the usual examples are $T = \mathbb{R}^+$ and $T = [a, b]$.

The index t represents time, and X_t is the "state" of the process at time t .

The state space is often \mathbb{R} (the real-valued process).

For every fixed $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega) (t \in T)$, is a sample function of the process.

2.8 The Poisson process

A Poisson process is the stochastic process in which events occur continuously and independently of one another.

The Poisson process is:

- a collection $\{X(t) : t \geq 0\}$ of random variables;
- where $X(t)$ is the number of events that have occurred up to time t .

The number of events between time a and time b is given as

$X(b) - X(a)$ and has a Poisson distribution.

Each realization of the process $\{X(t)\}$ is a non-negative integer-valued step function that is non-decreasing.

3 Transmission Control Protocol

The main data protocol of the Internet is

TCP (Transmission Control Protocol).

It is designed to adapt to the various traffic conditions of the network:

- a TCP connection between a source and a destination progressively increases its transmission rate until it receives some indication that the capacity along its path in the network is almost fully utilized.
- when the capacity of the network cannot accommodate the traffic, the data rate of the connection is drastically reduced.

[1] *F. Guillemin, Ph. Robert, and B. Zwart, AIMD algorithms and exponential functionals, Ann. Appl. Probab. Vol. 14, No. 1 (2004), 90–117.*

A given connection has a variable W which gives the maximum number of packets that can be transmitted without receiving any acknowledgment from the destination.

The variable W is called the **congestion window size**.

If all the W packets are successfully transmitted, then W is increased by 1, so that W packets can be sent for the next round.

Otherwise W is divided by 2 (detection of congestion).

An algorithm can be described as follows:

$$W = \begin{cases} W + 1, & \text{if no loss among the } W \text{ packets,} \\ \lfloor \delta W \rfloor, & \text{otherwise} \end{cases}$$

where $\delta = 1/2$ and $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$.

Via simulation, it was derived an asymptotic estimate for a **constant loss rate α** .

Let W_n^α denote the congestion window size over the n th RTT (Round Trip Time) interval, i.e. the total number of packets sent during this time interval.

The evolution of the process is given by

$$W_{n+1}^\alpha = \begin{cases} W_n^\alpha + 1, & \text{if none of } W \text{ packets is lost,} \\ \max(\lfloor \delta W_n^\alpha \rfloor, 1), & \text{otherwise.} \end{cases}$$

In the non-correlated case, each packet has a probability $p = 1 - e^{-\alpha}$ to be lost.

Let t_n^α denotes the index of n th packet which is lost. The independence assumption implies that the sequence

$$t_{n+1}^\alpha - t_n^\alpha \approx \frac{1}{\alpha} E_n \quad (\alpha \approx 0) ,$$

Where E_n is exponentially distributed with parameter 1. Asymptotically, the loss process can thus be described as a Poisson process. When a packet loss occurs several packets are also lost during the following RTT intervals.

Asymptotically, at the packet level, the loss process can thus be described as a Poisson process with clumps, i.e. a standard Poisson process with "clouds" around each of its points.

[2] J. BERTOIN , PH. BIANE, M. YOR, *Poissonian exponential functionals, q -series, q -integrals, and the moment problem for log-normal distributions*, Proceedings Stochastic Analysis, Ascona, 2002.

3.1 The invariant distributions

Let us consider random variables:

- (i) E_0 is an exponentially distributed with parameter 1;
- (ii) X_1 is independent of E_0 such that $\mathbb{P}(X_1 > 0) = 1$;

For $\beta \in (0, 1)$, we investigate the random variable I independent of E_0 and X_1 which is the solution to the equation

$$(1) \quad I \stackrel{\text{dist.}}{=} \beta^{X_1} I + E_0 .$$

By iterating the relation (1), the variable I can be represented as

$$I = \sum_{n=0}^{\infty} \beta^{S_n} E_n, \quad \text{where} \quad S_n = \sum_{k=1}^n X_k,$$

and (E_n) is a sequence of exponentially distributed random variables with parameter 1. Also, from (1), we can prove that

$$\mathcal{L}(I^n) = \frac{n}{1 - \mathcal{L}(\beta^n X_1)} \mathcal{L}(I^{n-1}) \quad (n \in \mathbb{N}).$$

Then

$$\mathcal{L}(e^{-\lambda I}) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \mathcal{L}(I^n) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\prod_{k=1}^n (1 - \mathcal{L}(\beta^k X_1))}.$$

If the random variable X_1 has a rational generating function, i.e. there exist two polynomials P and Q such that $E(z^{X_1}) = P(z)/Q(z)$.

Then

$$1 - E(z^{X_1}) = (1 - z) \frac{\prod_{j=1}^N (1 - b_j z)}{\prod_{i=1}^M (1 - a_i z)} .$$

Hence the following representation for the Laplace transform of random variable I can be established:

$$\mathcal{L}\left(e^{-\lambda I}\right) = \sum_{n=0}^{\infty} \frac{(a_1\beta; \beta)_n \cdots (a_M\beta; \beta)_n}{(b_1\beta; \beta)_n \cdots (b_N\beta; \beta)_n} \frac{(-\lambda)^n}{(\beta; \beta)_n} .$$

The last expression for the Laplace transform can be transformed so that it can be expressed as a q -hypergeometric functions. This suggests that q -calculus is the natural setting to study the density of exponential functionals.

3.2 The shifted geometric distribution

Let us consider the case when has a shifted geometric distribution

$$P(X_1 = n) = a^{n-1}(1 - a) \quad (0 < a < 1, n \in \mathbb{N})$$

The generating function is

$$E(z^{X_1}) = \frac{(1 - a)z}{1 - az} \quad \Rightarrow \quad 1 - E(z^{X_1}) = \frac{1 - z}{1 - az} \quad (|z| \leq 1).$$

If $a \notin \{\beta^k : k \in \mathbb{N}\}$, then

$$\mathcal{L}(e^{-\lambda I}) = \sum_{n=0}^{\infty} \frac{(a\beta; \beta)_n}{(\beta; \beta)_n} (-\lambda)^n.$$

Thank You for your attention!

THE END