Recent results on generalized inverses

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1. Introduction

Let X, Y, Z be Hilbert spaces, and let $\mathcal{L}(X,Y)$ denote the set of all linear bounded operators from X to Y. We abbreviate $\mathcal{L}(X) = \mathcal{L}(X,X)$. For $A \in \mathcal{L}(X,Y)$ we denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, the null-space and the range of A. An operator $B \in \mathcal{L}(Y,X)$ is an inner inverse of A, if $ABA = A$ holds. In this case A is inner invertible, or relatively regular. It is well-known that A is inner invertible if and only if $\mathcal{R}(A)$ is closed in Y. The Moore-Penrose inverse of $A \in \mathcal{L}(X,Y)$ is the operator $X \in \mathcal{L}(Y,X)$ which satisfies the Penrose equations

$$(1)\; AXA = A, \;\; (2)\; XAX = X, \;\; (3)\; (AX)^* = AX, \;\; (4)\; (XA)^* = XA.$$

The Moore-Penrose inverse of A exists if and only if $\mathcal{R}(A)$ is closed in Y. If the Moore-Penrose inverse of A exists, then it is unique, and it is denoted by $A^\dagger$.

If $\theta \subset \{1,2,3,4\}$, and X satisfies the equations (i) for all $i \in \theta$, then X is an θ-inverse of A. The set of all θ-inverses of A is denoted by $A\{\theta\}$. If $\mathcal{R}(A)$ is closed, then $A\{1,2,3,4\} = \{A^\dagger\}$.

Lemma 1.1. *Let* $A \in \mathcal{L}(X, Y)$ *have a closed range. Then* A *has the matrix decomposition with respect to the orthogonal decompositions of spaces* $X = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$ *and* $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*)$*:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 *is invertible. Moreover,*

$$A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

Lemma 1.2. *Let $A \in \mathcal{L}(X, Y)$ have a closed range. Let X_1 and X_2 be closed and mutually orthogonal subspaces of X, such that $X = X_1 \oplus X_2$. Let Y_1 and Y_2 be closed and mutually orthogonal subspaces of Y, such that $Y = Y_1 \oplus Y_2$. Then the operator A has the following matrix representations with respect to the orthogonal sums of subspaces $X = X_1 \oplus X_2 = \mathcal{R}(A^*) \oplus \mathcal{N}(A)$, and $Y = \mathcal{R}(A) \oplus \mathcal{N}(A^*) = Y_1 \oplus Y_2$:*

(a)

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $D = A_1 A_1^* + A_2 A_2^*$ maps $\mathcal{R}(A)$ into itself and $D > 0$ (meaning $D \geq 0$ invertible). Also,

$$A^\dagger = \begin{bmatrix} A_1^* D^{-1} & 0 \\ A_2^* D^{-1} & 0 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \to \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix},$$

where $D = A_1^* A_1 + A_2^* A_2$ maps $\mathcal{R}(A^*)$ into itself and $D > 0$ $\big($meaning $D \geq 0$ invertible$\big)$. Also,

$$A^\dagger = \begin{bmatrix} D^{-1} A_1^* & D^{-1} A_2^* \\ 0 & 0 \end{bmatrix}.$$

Here A_i denotes different operators in any of these two cases.

Let H be Hilbert spaces.

An operator $P \in \mathcal{L}(H)$ is idempotent if $P^2 = P$.

An operator $Q \in \mathcal{L}(H)$ is called orthogonal projection if $Q = Q^2 = Q^*$, where Q^* denotes the adjoint operator of Q.

A bounded linear operator $T \in \mathcal{L}(H)$ is Drazin invertible if and only if T has finite index, which is equivalent to the fact that 0 is a finite order pole of the resolvent operator $R_\lambda(T) = (\lambda I - T)^{-1}$, say of order k. In such case, $ind(T) = k$ and 0 is not the accumulation point of $\sigma(T)$.

For $T \in \mathcal{L}(H)$, the Drazin inverse T^{d} of T is unique if it exists and $(T^*)^{\mathsf{d}} = (T^{\mathsf{d}})^*$.

The Drazin invertibility of an operator in $\mathcal{L}(H)$ is similarly invariant, i.e. if T is Drazin invertible and $S \in \mathcal{L}(H)$ is an invertible operator, then $S^{-1}TS$ is Drazin invertible and $(S^{-1}TS)^{\mathsf{d}} = S^{-1}T^{\mathsf{d}}S$.

Lemma 1.3. *If* $A \in \mathcal{L}(X)$ *and* $B \in \mathcal{L}(Y)$ *are Drazin invertible,* $C \in \mathcal{L}(Y,X)$ *and* $D \in \mathcal{L}(X,Y)$*, then*

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \text{ and } N = \begin{bmatrix} A & 0 \\ D & B \end{bmatrix}$$

are also Drazin invertible and

$$M^{\mathsf{d}} = \begin{bmatrix} A^{\mathsf{d}} & S \\ 0 & B^{\mathsf{d}} \end{bmatrix}, \quad N^{\mathsf{d}} = \begin{bmatrix} B^{\mathsf{d}} & 0 \\ S & A^{\mathsf{d}} \end{bmatrix}, \tag{1}$$

where $S = \sum_{n=0}^{\infty}(A^{\mathsf{d}})^{n+2}CB^nB^{\pi} + \sum_{n=0}^{\infty}A^{\pi}A^nC(B^{\mathsf{d}})^{n+2} - A^{\mathsf{d}}CB^{\mathsf{d}}$.

Lemma 1.4. *Let* $M \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ *have the operator matrix form*

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}. \tag{2}$$

If two of the elements M, A *and* C *are Drazin invertible, then the third element is also Drazin invertible. In particular, if* $B = 0$*, then* M *is Drazin invertible if and only if* A *and* C *are Drazin invertible.*

Lemma 1.5. *Let* $M \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K})$ *have the operator matrix form*

$$M = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}. \tag{3}$$

Then M *is Drazin invertible if and only if* AB *(or* BA*) is Drazin invertible. In this case,*

$$M^{\mathsf{d}} = \begin{pmatrix} 0 & (AB)^{\mathsf{d}}A \\ B(AB)^{\mathsf{d}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A(BA)^{\mathsf{d}} \\ (BA)^{\mathsf{d}}B & 0 \end{pmatrix}.$$

Let $\mathbb{C}^{m\times n}$ be the set of $m \times n$ complex matrices. By $rank(A)$, $A^\top$, A^*, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ we denote the rank, transpose, conjugate transpose, range (column space) and null space, respectively, of $A \in \mathbb{C}^{m\times n}$.
If A is a complex matrix, then the smallest non-negative integer k, which satisfies $\mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)$, is called the index of A, denoted by $\mathrm{ind}(A)$. If $\mathrm{ind}(A) = 1$, then there exists the unique matrix A^g which satisfies the equations:

$$AA^gA = A,\ A^gAA^g = A^g,\ AA^g = A^gA.$$

The matrix A is the group inverse of A. Moreover, $\mathrm{ind}(A) = 0$ if and only if A is invertible, and ind this case $A^{-1} = A^g$.

Definition 1.1. *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. If a matrix $X \in \mathbb{C}^{n\times m}$ satisfies*

$$XAX = X, \quad \mathcal{R}(X) = T, \quad \mathcal{N}(X) = S,$$

then X is called the outer inverse or generalized inverse of A, and the notation $X = A^{(2)}_{T,S}$ is common.

The main characterization of $A^{(2)}_{T,S}$-generalized inverse is given as follows.

Lemma 1.6. *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. Then A has an outer inverse X such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if $AT \oplus S = \mathbb{C}^m$, and in this case $X = A^{(2)}_{T,S}$ is unique.*

We also need the following results.

Lemma 1.7. *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. In addition, suppose $G \in \mathbb{C}^{n\times m}$ such that $\mathcal{R}(G) = T$ and $\mathcal{N}(G) = S$. If A has an outer inverse $A^{(2)}_{T,S}$, then $\mathrm{ind}(AG) = \mathrm{ind}(GA) = 1$. Further, we have*

$$A^{(2)}_{T,S} = (GA)^g G = G(AG)^g. \tag{4}$$

Lemma 1.8. *If A satisfies the conditions of Lemma 1.7, then*

$$rank(AG) = rank(GA) = rank(G).$$

If A is square and invertible, then the condition number of A is defined as $k(A) = \|A\| \cdot \|A^{-1}\|$, where $\| \cdot \|$ is some matrix norm. The study of condition numbers is important in the theory of stability of linear systems. If A is rectangular (or even square and singular), then we do not have the condition number of A in the previous sense. But still, we have some generalized inverse of A, say A^-. Now, the "generalized" condition number of A related to A^- is defined as $\|A\| \cdot \|A^-\|$. Generalized condition numbers have applications in studying singular linear systems.
The following result is known as the Schur decomposition theorem.

Lemma 1.9. (Schur decomposition) *If $A \in \mathbb{C}^{n\times n}$, then there exists an unitary $U \in \mathbb{C}^{n\times n}$ such that*

$$U^* AU = T = D + N$$

where $D = diag(\lambda_1, \ldots, \lambda_n)$ and $N \in C^{n\times n}$ is strictly upper triangular. Furthermore, U can be chosen so that the eigenvalues λ_i appear in any order along the diagonal.

Let (X, ρ) be a complete metric space. A map $T : X \mapsto X$ such that for some constant $\lambda \in (0,1)$ and for every $x, y \in X$

$$d(Tx, Ty) \leq \lambda \cdot \max\Big\{\rho(x,y), \rho(x,Tx), \rho(y,Ty), \rho(x,Ty), \rho(y,Tx)\Big\} \quad (5)$$

is called *quasicontraction*.
Let E be a real Banach space and P a subset of E. P is called a cone if and only if:

(i) P is closed, nonempty, and $P \neq \{0\}$,

(ii) $a, b \in R, a, b \geq 0, x, y \in P \Longrightarrow ax + by \in P$

(iii) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$; we shall write $x \ll y$ if $y - x \in \text{int } P$, where int P denotes the interior of P.
The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \quad \text{implies} \quad \|x\| \leq K\|y\|.$$

Definition 1.2. *Let X be a nonempty set. Suppose the mapping $d : X \times X \mapsto E$ satisfies*

(d1) $0 < d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$*;*

(d2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

For $F \subset E$, we define $\delta(F) = \sup\{\|x\| : x \in F\}$.

Let x_n be a sequence in X, and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x) \ll c$, then x_n is said to be convergent, and x_n converges to x, and we denote this by $\lim_n x_n = x$, or $x_n \to x$, $(n \to \infty)$. If for every $c \in E$ with $0 \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n, m > n_0, d(x_n, x_m) \ll c$, then x_n is called a Cauchy sequence in X. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Let us recall that if P is a normal cone then $x_n \in X$ converges to $x \in X$ if and only if $d(x_n, x) \to 0, n \to \infty$. Further, $x_n \in X$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0, n, m \to \infty$.

If E is a real Banach space with cone P and if $a \leq \lambda a$ where $a \in P$ and $0 < \lambda < 1$, then $a = 0$. The condition $a \leq \lambda a$ means that $\lambda a - a \in P$, i.e., $-(1-\lambda)\,a \in P$. Since $a \in P$ and $1 - \lambda > 0$, then also $(1-\lambda)\,a \in P$. Thus we have $(1-\lambda)\,a \in P \cap (-P) = \{0\}$, hence $a = 0$.
Let (X, d) be a cone metric space. Then:

If $o \leq x \leq y$, and $a \geq 0$, then it is easy to prove that $0 \leq ax \leq ay$.

If $0 \leq x_n \leq y_n$ for each $n \in \mathbb{N}$, and $\lim_n x_n = x$, $\lim_n y_n = y$, then $0 \leq x \leq y$.

In the next definition we define quasi contraction on cone metric space. Such a mapping is a generalization of Ćirić's quasi contraction.

Definition 1.3. *Let (X, d) be a cone metric space. A map $f : X \mapsto X$ such that for some constant $\lambda \in (0, 1)$ and for every $x, y \in X$, there exists $u \in C(f, x, y) \equiv \Big\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\Big\}$, such that*

$$d(fx, fy) \leq \lambda \cdot u, \tag{6}$$

is called quasi contraction.

If $f : X \mapsto X$, and $n \in \mathbb{N}$, we set

$$O(x; n) = \Big\{x, fx, f^2x, ..., f^nx\Big\}, \text{ and } O(x; \infty) = \Big\{x, fx, f^2x, ...\Big\}.$$

2. Reverse order law for the Moore-Penrose inverse

If S is a semigroup with the unit 1, and if $a, b \in S$ are invertible, then the equality $(ab)^{-1} = b^{-1}a^{-1}$ is called the reverse order law for the ordinary inverse. It is well-known that the reverse order law does not hold for various classes of generalized inverses.

In this section we present new results related to the reverse order law for the Moore-Penrose inverse of operators on Hilbert spaces. Some finite dimensional results are extended to infinite dimensional settings (Y. Tian, *Using rank formulas to characterize equalities for Moore-Penrose inverses of matrix products*, Appl. Math. Comput. 147 (2004), 581–600). The results of this section are proved in the paper:

D.S. Djordjević, N.Č. Dinčić, *Reverse order law for the Moore–Penrose inverse*, J. Math. Anal. Appl. 361 (1) (2010), 252-261.

Theorem 2.1. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$ and $B \in \mathcal{L}(X, Y)$ be such that A, B, AB have closed ranges. Then the following statements are equivalent:*

(a) $ABB^\dagger A^\dagger AB = AB$;

(b) $B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger$;

(c) $A^\dagger ABB^\dagger = BB^\dagger A^\dagger A$;

(d) $A^\dagger ABB^\dagger$ *is an idempotent;*

(e) $BB^\dagger A^\dagger A$ *is an idempotent;*

(f) $B^\dagger (A^\dagger ABB^\dagger)^\dagger A^\dagger = B^\dagger A^\dagger$;

(g) $(A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A$;

Now we prove the following result.

Theorem 2.2. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be such that A, B, AB have closed ranges. Then the following statements hold:*

(a) $AB(AB)^\dagger = ABB^\dagger A^\dagger \Leftrightarrow A^*AB = BB^\dagger A^*AB \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1, 2, 3\}$;

(b) $(AB)^\dagger AB = B^\dagger A^\dagger AB \Leftrightarrow ABB^* = ABB^*A^\dagger A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^\dagger A^\dagger \in (AB)\{1, 2, 4\}$;

(c) *The following statements are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$;

(2) $AB(AB)^\dagger = ABB^\dagger A^\dagger$ *and* $(AB)^\dagger AB = B^\dagger A^\dagger AB$;

(3) $A^*AB = BB^\dagger A^*AB$ *and* $ABB^* = ABB^*A^\dagger A$;

(4) $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ *and* $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

We also prove the following result.

Theorem 2.3. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be such that A, B, AB have closed ranges. Then we have:*

(a) $AB(AB)^{\dagger}A = ABB^{\dagger} \Leftrightarrow A^*ABB^{\dagger} = BB^{\dagger}A^*A \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B) \Leftrightarrow B^{\dagger}A^{\dagger} \in (AB)\{1, 2, 3\}$;

(b) $B(AB)^{\dagger}AB = A^{\dagger}AB \Leftrightarrow A^{\dagger}ABB^* = BB^*A^{\dagger}A \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow B^{\dagger}A^{\dagger} \in (AB)\{1, 2, 4\}$;

(c) *The following three statements are equivalent:*

(1) $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$;

(2) $AB(AB)^{\dagger}A = ABB^{\dagger}$ *and* $B(AB)^{\dagger}AB = A^{\dagger}AB$;

(3) $A^*ABB^{\dagger} = BB^{\dagger}A^*A$ *and* $A^{\dagger}ABB^* = BB^*A^{\dagger}A$.

Theorem 2.4. *Let* X, Y, Z *be Hilbert spaces, and let* $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ *be such that* A, B, AB *have closed ranges. The following statements hold.*

(a) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger \Leftrightarrow B^\dagger(ABB^\dagger)^\dagger = B^\dagger A^\dagger \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$.

(b) $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A \Leftrightarrow (A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger \Leftrightarrow \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$.

(c) *The following three statements are equivalent:*

(1) $(AB)^\dagger = B^\dagger A^\dagger$;

(2) $(ABB^\dagger)^\dagger = BB^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger = B^\dagger A^\dagger A$;

(3) $B^\dagger(ABB^\dagger)^\dagger = B^\dagger A^\dagger$ *and* $(A^\dagger AB)^\dagger A^\dagger = B^\dagger A^\dagger$.

Theorem 2.5. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be such that A, B, AB have closed ranges. Then we have:*

(a) $B^\dagger = (AB)^\dagger A \Leftrightarrow \mathcal{R}(B) = \mathcal{R}(A^* AB)$.

(b) $A^\dagger = B(AB)^\dagger \Leftrightarrow \mathcal{R}(A^*) = \mathcal{R}(BB^* A^*)$.

We need the following auxiliary result.

Lemma 2.1. *Let X, Y be Hilbert spaces, let $C \in \mathcal{L}(X, Y)$ have a closed range, and let $D \in \mathcal{L}(Y)$ be Hermitian and invertible. Then $\mathcal{R}(DC) = \mathcal{R}(C)$ if and only if $[D, CC^\dagger] = 0$.*

Theorem 2.6. *Let X, Y, Z be Hilbert spaces, and let $A \in \mathcal{L}(Y, Z)$, $B \in \mathcal{L}(X, Y)$ be such that A, B, AB have closed ranges. Then we have:*

(a) $(AB)^\dagger = (A^\dagger AB)^\dagger A^\dagger \Leftrightarrow \mathcal{R}(AA^* AB) = \mathcal{R}(AB)$;

(b) $(AB)^\dagger = B^\dagger (ABB^\dagger)^\dagger \Leftrightarrow \mathcal{R}(B^* B(AB)^*) = \mathcal{R}((AB)^*)$.

3. The Drazin invertibility of the difference and the sum of two idempotent operators

In this section some equivalents are established of the Drazin invertibility of differences and sums of idempotent operators on a Hilbert space, using the spectral theory of linear operators. The results of this section are presented in the paper:

D. S. Cvetković-Ilić and C. Y. Deng, *The Drazin invertibility of the difference and the sum of two idempotent*, J. Comput. Appl. Math. doi:10.1016/j.cam.2009.09.028.

In the recent years, a number of researchers have considered questions concerning the idempotents. The authors obtain results for the Drazin invertibility of the sum and difference of the idempotents analogous to those of Koliha and Rakočević (J. J. Koliha, V. Rakočević and I. Strakraba, *The difference and sum of projectors*, Linear Algebra and its Applications, 388 (2004), 279-288; J. J. Koliha and V. Rakočević, *Invertibility of the sum of idempotents*, Linear and Multilinear Algebra 50 (2002), 285-292; J. J. Koliha and V. Rakočević, *Invertibility of the difference of idempotents*, Linear and Multilinear Algebra 51 (2003), 97-110) in the case of ordinary invertibility.

The following theorem is a well-known result which we state for the sake of completeness:

Theorem 3.1. *Let P be an idempotent and Q be an orthogonal projection in $\mathcal{L}(H)$. The following statements are equivalent:*

(1) PQ *is Drazin invertible,*
(2) QP *is Drazin invertible,*
(3) PQP *is Drazin invertible,*
(4) QPQ *is Drazin invertible,*
(5) P^*Q *is Drazin invertible,*
(6) QP^* *is Drazin invertible,*
(7) P^*QP^* *is Drazin invertible,*
(8) QP^*Q *is Drazin invertible.*

If in Theorem 3.1 we replace P and Q by $I-P$ and $I-Q$, respectively, we get the following results:

Corollary 3.1. *Let P be an idempotent and Q be an orthogonal projection in $\mathcal{L}(H)$. The following statements are equivalent:*

(1) $(I-P)(I-Q)$ *is Drazin invertible,*
(2) $(I-Q)(I-P)$ *is Drazin invertible,*
(3) $(I-P)(I-Q)(I-P)$ *is Drazin invertible,*
(4) $(I-Q)(I-P)(I-Q)$ *is Drazin invertible,*
(5) $(I-P)^*(I-Q)$ *is Drazin invertible,*
(6) $(I-Q)(I-P)^*$ *is Drazin invertible,*
(7) $(I-P)^*(I-Q)(I-P)^*$ *is Drazin invertible,*
(8) $(I-Q)(I-P)^*(I-Q)$ *is Drazin invertible.*

Lemma 3.1. *Let $A, B \in \mathcal{L}(H)$. Then $I-AB$ is Drazin invertible if and only if $I-BA$ is Drazin invertible.*

Let us remark that the analogue result as in the next theorem concerning ordinary invertibility is proved in Theorem 3.2 (J. J. Koliha and V. Rakočević, *Invertibility of the difference of idempotents*, Linear and Multilinear Algebra 51 (2003),).

Theorem 3.2. *Let P, Q be idempotents in $\mathcal{L}(H)$. Then $P - Q$ is Drazin invertible if and only if $I - PQ$ and $P + Q - PQ$ are Drazin invertible.*

Corollary 3.2. *Let P, Q be idempotents in $\mathcal{L}(H)$. The following statements are equivalent:*
(1) $I - PQ$ *is Drazin invertible,*
(2) $P - PQ$ *is Drazin invertible,*
(3) $I - PQP$ *is Drazin invertible,*
(4) $P - PQP$ *is Drazin invertible,*
(5) $I - QP$ *is Drazin invertible,*
(6) $Q - QP$ *is Drazin invertible,*
(7) $I - QPQ$ *is Drazin invertible,*
(8) $Q - QPQ$ *is Drazin invertible.*

As before, if in Corollary 3.4 we replace P and Q by $I-P$ and $I-Q$, respectively, we have the following result:

Corollary 3.3. *Let P,Q be idempotents in $\mathcal{L}(H)$. The following statements are equivalent:*

(1) $P+Q-PQ$ *is Drazin invertible,*
(2) $Q-PQ$ *is Drazin invertible,*
(3) $P+(I-P)Q-(I-P)QP$ *is Drazin invertible,*
(4) $(I-P)Q(I-P)$ *is Drazin invertible,*
(5) $P+Q-QP$ *is Drazin invertible,*
(6) $P-QP$ *is Drazin invertible,*
(7) $Q+(I-Q)P-(I-Q)PQ$ *is Drazin invertible,*
(8) $(I-Q)P(I-Q)$ *is Drazin invertible.*

Corollary 3.4. (1) *Let P and Q be orthogonal projections in $\mathcal{L}(H)$. Then the conditions in Theorem 3.3, Corollary 3.2 and Corollary 3.3 are all equivalent to the fact that $P+Q$ is Drazin invertible.*

(2) *Let $P,Q\in\mathcal{L}(H)$ be idempotents. Then $P-Q$ is Drazin invertible if and only if one of the conditions from the Corollary 3.2 and one of the conditions from the Corollary 3.3 hold.*

Lemma 3.2. *Let $P \in \mathcal{L}(H)$ be an idempotent and $Q \in \mathcal{L}(H)$ be an orthogonal projection. Then P and Q have the following operator matrices*

$$P = \begin{pmatrix} I & & & P_{14} & P_{15} & P_{16} \\ & I & & P_{24} & P_{25} & P_{26} \\ & & I & P_{34} & P_{35} & P_{36} \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}, \tag{7}$$

$$Q = \begin{pmatrix} I & & & & & \\ & 0 & & & & \\ & & Q_1 & Q_1^{\frac{1}{2}}(I-Q_1)^{\frac{1}{2}}D & & \\ & & D^*Q_1^{\frac{1}{2}}(I-Q_1)^{\frac{1}{2}} & D^*(I-Q_1)D & & \\ & & & & 0 & \\ & & & & & I \end{pmatrix} \tag{8}$$

with respect to the space decomposition $H = \sum_{i=1}^{6} H_i$, where P_{ij} is an operator from H_j into H_i, $1 \leq i \leq 3$, $4 \leq j \leq 6$, Q_1 is a positive contraction on H_3, 0 and 1 are not the eigenvalues of Q_1, D is a unitary operator from H_4 onto H_3 and the entries omitted in the formula (7) and (8) are zero.

4. Condition number related to the outer inverse of a complex matrix

In this section we obtain the formula for computing the condition number of a complex matrix, which is related to the outer generalized inverse of a given matrix. We use the Schur decomposition of a matrix. This results are proved in the paper

D. Mosić, D.S. Djordjević, *Condition number related to the outer inverse of a complex matrix*, Appl. Math. Comput. doi:10.1016/j.amc.2009.09.023.

These results generalize some early work (H. Diao, M. Qin, Y. Wei, *Condition numbers for the outer inverse and constrained singular linear system*, Appl. Math. Comput. 174 (2006) 588–612; Y. Wei, N. Zhang, *Condition number with generalized inverse $A^{(2)}_{T,S}$ and constrained linear systems*, J. Comput. Appl. Math. 157 (2003) 57–72.), because of the well-posed properties of the Schur decomposition. Some results are established for the condition number of the generalized inverse and the generalized inverse solution of a linear system, using a special norm called PQ-norm which depends on the Jordan canonical form of A. The computation of the Jordan canonical form is an ill-posed problem.

Let $A \in \mathbb{C}^{n\times n}$ satisfies the following condition:

$$rank(A^k) = r, \quad \mathrm{ind}(A) = k, \quad \mathcal{R}(A^k) = \mathcal{R}(A^{k^*}). \tag{9}$$

We prove the following result.

Theorem 4.1. *Let A, G, T and S be the same as in Lemma 1.7, $p = rank(AG)$, $\mathcal{R}(AG) = \mathcal{R}((AG)^*)$ and $\mathcal{R}(GA) = \mathcal{R}((GA)^*)$. Then we have*

$$A = V \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} U^*, \qquad G = U \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} V^*$$

$$A^{(2)}_{T,S} = U \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*, \tag{10}$$

where U and V are unitary matrices, A_1 and G_1 are nonsingular matrices.

In this section we consider the following linear system

$$Ax = b, \quad x \in T,$$

where $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. The generalized $A^{(2)}_{T,S}$-inverse solution x has the form

$$x = A^{(2)}_{T,S}b.$$

If F is a continuously differentiable function $F : \mathbb{C}^{m\times n} \times \mathbb{C}^m \longrightarrow \mathbb{C}^n, (A, x) \longmapsto F(A, x)$, then the absolute condition number of F at x is the scalar $\|F'(x)\|$. The relative condition of F at x is $\frac{\|F'(x)\|\|x\|}{\|y\|}$.
The following operator:

$$F : \mathbb{C}^{m\times n} \times \mathbb{C}^m \longrightarrow \mathbb{C}^n$$

$$(A, b) \longmapsto F(A, b) = A^{(2)}_{T,S}b = x$$

is differentiable function, if the perturbation E in A fulfils the following condition:

$$\mathcal{R}(E) \subseteq AT, \quad \mathcal{R}(E^*) \subseteq A^*S^\top. \tag{11}$$

It is easy to verify that (11) is equivalent to

$$AA^{(2)}_{T,S}E = E, \quad EA^{(2)}_{T,S}A = E. \tag{12}$$

We get the explicit formula for the condition number of the generalized $A_{T,S}^{(2)}$-inverse solution by means of the 2-norm and Frobenius norm.

Theorem 4.2. *Let A, G, T and S be the same as in Lemma 1.7, $p = rank(AG)$, $\mathcal{R}(AG) = \mathcal{R}((AG)^*)$ and $\mathcal{R}(GA) = \mathcal{R}((GA)^*)$. If the perturbation E in A fulfills the condition (11), then the absolute condition number of the generalized $A_{T,S}^{(2)}$-inverse solution of a linear system, with the norm*

$$\|[\alpha A, \beta b]\|_{U,Q}^{(F)} = \sqrt{\alpha^2\|A\|_F^2 + \beta^2\|b\|_2^2}$$

on the data (A, b), and the norm $\|x\|_2$ on the solution, is given by

$$C = \|A_{T,S}^{(2)}\|_2\sqrt{\frac{1}{\beta^2} + \frac{\|x\|_2^2}{\alpha^2}},$$

where $Q = \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix}$ and U is the same matrix as in (10).

If E satisfies the condition (11), then the 2-norm relative condition number of the generalized inverse $A_{T,S}^{(2)}$ is defined as

$$Cond(A) = \lim_{\epsilon \to 0^+} \sup_{\|E\|_2 \le \epsilon \|A\|_2} \frac{\|(A+E)_{T,S}^{(2)} - A_{T,S}^{(2)}\|_2}{\epsilon \|A_{T,S}^{(2)}\|_2}$$

and the corresponding condition number for the linear systems $Ax = b$ is defined as

$$Cond(A,b) = \lim_{\epsilon \to 0^+} \sup_{\substack{\|E\|_2 \le \epsilon \|A\|_2 \\ \|f\|_2 \le \epsilon \|b\|_2}} \frac{\|(A+E)_{T,S}^{(2)}(b+f) - A_{T,S}^{(2)} b\|_2}{\epsilon \|A_{T,S}^{(2)} b\|_2}.$$

The level-2 condition number of the generalized $A_{T,S}^{(2)}$-inverse is defined as

$$Cond^{[2]}(A) = \lim_{\epsilon \to 0} \sup_{\|E\|_2 \le \epsilon \|A\|_2} \frac{|Cond(A+E) - Cond(A)|}{\epsilon Cond(A)}$$

and the level-2 corresponding condition number is defined as

$$Cond^{[2]}(A,b) = \lim_{\epsilon \to 0} \sup_{\substack{\|E\|_2 \le \epsilon \|A\|_2 \\ \|f\|_2 \le \epsilon \|b\|_2}} \frac{|Cond(A+E, b+f) - Cond(A,b)|}{\epsilon Cond(A,b)}.$$

First, we proved the following lemmas.

Lemma 4.1. *For $\hat{u}, \hat{v}$ in Theorem 2.1, there exists $S \in \mathbb{C}^{m\times n}$ such that*

$$S\hat{v} = -\hat{u}, \quad \|S\|_2 = 1,$$

where S fulfills condition (11).

Lemma 4.2. *Let A, G, T and S be the same as in Lemma 1.7, $p = rank(AG)$, $\mathcal{R}(AG) = \mathcal{R}((AG)^*)$ and $\mathcal{R}(GA) = \mathcal{R}((GA)^*)$. If $\epsilon \to 0$, then*

$$\max_{\|E\|_2 \le \epsilon \|A\|_2} \left| \|(A+E)^{(2)}_{T,S}\|_2 - \|A^{(2)}_{T,S}\|_2 \right| = \epsilon \|A^{(2)}_{T,S}\|_2 Cond(A) + \mathcal{O}(\epsilon^2),$$

provided that E fulfills the condition (11).

The following results show that for the generalized $A_{T,S}^{(2)}$-inverse for solving a linear system, the sensitivity of the condition number is approximately given by the condition number itself.

Corollary 4.1. *Let* A, G, T *and* S *be the same as in Lemma 1.7,* $p = rank(AG)$, $\mathcal{R}(AG) = \mathcal{R}((AG)^*)$ *and* $\mathcal{R}(GA) = \mathcal{R}((GA)^*)$. *If the perturbation* E *in* A *fulfills the condition* (11)*, then the level-2 condition number*

$$Cond^{[2]}(A) = \lim_{\epsilon \to 0} \sup_{\|E\|_2 \le \epsilon \|A\|_2} \frac{|Cond(A+E) - Cond(A)|}{\epsilon Cond(A)} \tag{13}$$

satisfies

$$|Cond^{[2]}(A) - Cond(A)| \le 1. \tag{14}$$

Corollary 4.2. *Let A, G, T and S be the same as in Lemma 1.7, $p = rank(AG)$, $\mathcal{R}(AG) = \mathcal{R}((AG)^*)$ and $\mathcal{R}(GA) = \mathcal{R}((GA)^*)$. If the perturbation E in A fulfills the condition (11), then the level-2 condition number of linear systems $Ax = b$, $x \in T$,*

$$Cond^{[2]}(A,b) = \lim_{\epsilon \longrightarrow 0} \sup_{\substack{\|E\|_2 \le \epsilon \|A\|_2 \\ \|f\|_2 \le \epsilon \|b\|_2}} \frac{|Cond(A+E, b+f) - Cond(A,b)|}{\epsilon Cond(A,b)} \quad (15)$$

satisfies

$$\frac{Cond(A,b)}{(1+\zeta)^2} - \frac{1}{1+\zeta} \le Cond^{[2]}(A,b) \le 3Cond(A,b) + 2, \quad (16)$$

where $\zeta = \frac{\|b\|_2}{\|AA^{(2)}_{T,S}b\|_2}$.

Now we present a structured perturbation of the generalized inverse $A^{(2)}_{T,S}$ by means of 2–norm. The notation $|A| \leq |B|$ means that $|a_{i,j}| \leq |b_{i,j}|$ for $A = (a_{i,j})$ and $B = (b_{i,j})$.

Theorem 4.3. *Let A, G, T and S be the same as in Lemma 1.7, $p = rank(AG)$, $\mathcal{R}(AG) = \mathcal{R}((AG)^*)$ and $\mathcal{R}(GA) = \mathcal{R}((GA)^*)$. If $|V^*EU| \leq |V^*AU|$ and $\|A^{(2)}_{T,S}E\|_2 < 1$, then*

$$(A+E)^{(2)}_{T,S} = (I + A^{(2)}_{T,S}E)^{-1}A^{(2)}_{T,S},$$

where U and V are the same matrices as in (10).

5. Quasi-contraction on cone metric space

In this section we define and study quasi contraction on cone metric space. For such a mapping we prove a fixed point theorem. This results are presented in the paper

D. Ilić, V. Rakočević, *Quasi-contraction on cone metric space*, Applied Mathematics Letters, 22 (2009) 728-731.

Among other things, the authors generalize recent result of H. L. Guang and Z. Xian, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl **332** (2007), 1468–1476, and the main result of Lj. B. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc., **45** (1974), 267–273, is also recovered.

We start with the next auxiliary result.

Lemma 5.1. *Let (X, d) be a cone metric space and P be a normal cone, Let $f : X \mapsto X$ be a quasi contraction. Then, there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$,*

$$\delta(O(x;n)) = \max\left\{\left\|d(x, f^l x)\right\|, \left\|d(f^i x, f^j x)\right\| : 1 \leq l \leq n, 1 \leq i, j \leq n_0\right\} \tag{17}$$

and

$$\begin{aligned} \delta(O(x,\infty)) \leq \max\Big\{&\frac{K}{1-K^2\lambda^{n_0}}\left\|d(x, f^{n_0+1}x)\right\|, \lambda K \delta(O(x;n_0)), \\ &\left\|d(x, f^l x)\right\| : 1 \leq l \leq n_0\Big\}. \end{aligned} \tag{18}$$

Theorem 5.1. *Let (X, d) be a complete cone metric space and P be a normal cone. Suppose the mapping $f : X \mapsto X$ is a quasi contraction. Then f has a unique fixed point in X and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point.*

Now, as a corollary, we get the main result of Guang and Z. Xian (*Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl **332** (2007), 1468–1476).

Corollary 5.1. *Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Suppose the mapping $f : X \mapsto X$ satisfies the contractive condition*

$$d(fx, fy) \leq \lambda d(x, y), \quad \textit{for all } x, y \in X \tag{19}$$

where $\lambda \in [0, 1)$ is constant. Then f has a unique fixed point in X and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point.

Let us remark that in Theorem 2.1, setting $E = \mathbb{R}$, $P = [0, \infty)$, $\|x\| = |x|, x \in E$, we get the well-know Ćirić's result (*A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273) for quasi-contraction.