Fuzzy and weighted automata:
determinization, state reduction, structural equivalence

– a short overview of our current research –

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**Theoretical Computer Science - From Foundation to Application,
Niš, Serbia, November 10, 2009**

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Fuzzy relations, fuzzy equivalences

Research on fuzzy relations

- our research in the **Algebraic theory of fuzzy automata and languages** required a firm background in the **Theory of fuzzy relations**
- our research has been mainly aimed to
  - **fuzzy equivalences** (fuzzy equivalence classes, fuzzy partitions)
  - **uniform fuzzy relations** (fuzzy functions, uniform relation morphisms, fuzzy homomorphisms of algebras)
  - **fuzzy quasi-orders** (aftersets, foresets)
  - **fuzzy relation equations**

Research on fuzzy equivalences

- **Fuzzy equivalences** have important applications in many fields, especially in fuzzy cluster analysis
- we have successfully used them in the state reduction of fuzzy automata
- we have also used them in the positional analysis of fuzzy networks
- we have also studied Nerode and Myhill fuzzy equivalences associated with fuzzy automata

Uniform fuzzy relations

Research on uniform fuzzy relations

- **uniform fuzzy relations** are a brand-new concept introduced in our recent paper
- they can be conceived as fuzzy equivalences which relate elements of two possibly different sets
- they are also a generalization of the concept of a function
- we have given some applications of uniform fuzzy relations in **approximate reasoning**, especially in **fuzzy control**
- we have shown that they are closely related to the **defuzzification problem**
- in algebra we have used them to define and study uniform relational morphisms and fuzzy homomorphisms
- in the theory of fuzzy automata we have applied them to study of **bisimulations** and structural equivalence between fuzzy automata
- we set ourself the task to transmit this concept, as well as other related concepts, to weighted automata

Fuzzy quasi-orders, fuzzy relation equations

Research on fuzzy quasi-orders

- **fuzzy quasi-orders** are another generalization of fuzzy equivalences
- they have important applications in fuzzy logic
- we study **aftersets** and **foresets** of fuzzy quasi-orders, as generalizations of fuzzy equivalence classes
- we have successfully applied them to state reduction of fuzzy automata

Research on fuzzy relation equations

- **fuzzy relation equations** have significant applications in many areas of the theory of fuzzy sets
- they are very important in approximate reasoning, fuzzy control, etc.
- we have established close relationships between state reduction of fuzzy recognizers and resolution of particular systems of fuzzy relation equations
- we have also studied fuzzy relation equations in the context of positional analysis of fuzzy networks

Articles on fuzzy relations

Articles on fuzzy relations

1. M. Ćirić, J. Ignjatović, S. Bogdanović, Fuzzy equivalence relations and their equivalence classes, Fuzzy Sets and Systems 158 (2007), 1295–1313.
2. M. Ćirić, J. Ignjatović, S. Bogdanović, Uniform fuzzy relations and fuzzy mappings, Fuzzy Sets and Systems 160 (2009) 1054–1081.
3. J. Ignjatović, M. Ćirić, S. Bogdanović, Fuzzy homomorphisms of algebras, Fuzzy Sets and Systems 160 (2009), 2345–2365.
4. M. Ćirić, J. Ignjatović, B. Šešelja, A. Tepavčević, Fuzzy quasi-orders, (in preparation).

Fuzzy automata and languages

Research on fuzzy automata and languages

- in the area of **fuzzy automata and languages** our research is mainly aimed on:
  - deterministic fuzzy recognizers, and determinization of fuzzy recognizers
  - state reduction methods
  - bisimulations and structural equivalence of fuzzy automata
  - fuzzy regular expressions and their conversion to fuzzy automata
- many our research themes also encroach in the theory of **weighted automata and formal power series**
- we have also given certain applications of fuzzy automata in the theory of **fuzzy discrete event systems**.

Deterministic fuzzy recognizers

Deterministic fuzzy recognizers

- a **(crisp) deterministic fuzzy recognizer** is a fuzzy recognizer such that
  - the fuzzy set of initial states is a crisp singleton set
  - the fuzzy transition relation is a crisp function
  - only the fuzzy set of terminal states is really fuzzy
- we have developed the general **Myhill-Nerode type theory** for fuzzy languages and deterministic fuzzy recognizers
- the underlying structure of truth values is very general – an **arbitrary set** with two distinguished elements $\mathbf{0}$ and $\mathbf{1}$
- $\mathbf{0}$ and $\mathbf{1}$ are needed to include crisp sets and languages as special cases of fuzzy sets and languages
- this general structure includes almost everything
  - all structures used for modelling membership values in the fuzzy set theory
  - semirings
  - strong bimonoids. . .

The minimal deterministic fuzzy recognizer

The minimal deterministic fuzzy recognizer

- we have proved that any fuzzy language possess the **minimal deterministic fuzzy recognizer** recognizing it
- if this minimal deterministic fuzzy recognizer of a fuzzy language is finite, then it is **unique up to an isomorphism**
- note that **minimal fuzzy recognizers are not unique up to an isomorphism**
- **construction of the minimal deterministic fuzzy recognizer**: using the concept of the **derivative automaton** of a fuzzy language
- states of the derivative automaton are **derivatives** of a fuzzy language
- we have given an **effective method for construction of the derivative automaton** of a fuzzy language
- it is based on simultaneous (parallel) construction of the derivative automata of the kernel (or cut) languages of this fuzzy language
- we have proved that a fuzzy language can be recognized by a deterministic finite recognizer if and only if it has a finite rank and all its kernel languages (or cut languages) are recognizable
- we have given a method for **minimization of deterministic fuzzy recognizers**

Determinization of fuzzy recognizers

Determinization of fuzzy recognizers

- **determinization of a fuzzy recognizer** is the problem of constructing, for a given fuzzy recognizer, a language equivalent deterministic fuzzy recognizer
- the **fuzzy subset construction** – the method for determinization of fuzzy recognizers developed by R. Bělohlávek (2002) and Y. M. Li and W. Pedrycz (2005)
- we have given the so-called **accessible fuzzy subset construction**
- it always gives a smaller recognizer than the fuzzy subset construction, and it can give a finite recognizer even if the fuzzy subset construction gives an infinite one
- the accessible fuzzy subset construction has been given in the context of fuzzy automata over complete residuated lattices
- it can be easily transmitted to fuzzy automata over lattice ordered monoids and weighted automata over semirings
- in this moment we are working on determinization of weighted automata over strong bimonoids (semirings where the distributivity is dropped)
- we are also working on other determinization methods which give even smaller recognizers, such as the **fuzzy transition set construction**, **Brzozovski's fuzzy canonization**, etc.

Articles on deterministic fuzzy recognizers

Articles on deterministic fuzzy recognizers

1. T. Petković, Varieties of fuzzy languages, Proceedings of the 1st International Conference on Algebraic Informatics, Aristotle University of Thessaloniki, Thessaloniki, 2005, pp. 197–205.
2. J. Ignjatović, M. Ćirić, S. Bogdanović, Determinization of fuzzy automata with membership values in complete residuated lattices, Information Sciences 178 (2008) 164–180.
3. J. Ignjatović, M. Ćirić, S. Bogdanović, T. Petković, Myhill-Nerode type theory for fuzzy languages and automata, Fuzzy Sets and Systems (2009), doi: 10.1016/j.fss.2009.06.007.
4. J. Ignjatović, M. Ćirić, Formal power series and regular operations on fuzzy languages, Information Sciences (accepted for publication).
5. M. Ćirić, M. Droste, J. Ignjatović, H. Vogler, Determinization of weighted finite automata over strong bimonoids, (in preparation).
6. Z. Jančić, J. Ignjatović, M. Ćirić, Determinization and canonization of weighted and fuzzy automata, (in preparation).

State reduction

State reduction of fuzzy automata

- the **state minimization problem** for nondeterministic and fuzzy automata is **computationally hard**
- instead of **state minimization** we rather consider **state reduction**
- the **state reduction methods** do not necessary give a minimal automaton, but they give a "reasonably" small automaton that can be constructed efficiently
- we have established close relationships between state reduction of a fuzzy recognizer and resolution to a particular system of fuzzy relation equations (the **general system**)
- we are interested in those solutions of this system which are fuzzy equivalences or fuzzy quasi-orders
- to obtain the best possible reduction of a fuzzy recognizer, we have to find as big a solution of the general system as possible
- the general system does not necessary have the greatest solution
- also, it may consist of infinitely many equations, and finding its nontrivial solutions may be a very difficult task
- for that reason we aim our attention to some instances of the general system, which have to be as general as possible, but they have to be easier to solve

Instances of the general system

Important instances of the general system

- we consider state reductions by means of
  - **right** and **left invariant** fuzzy equivalences (fuzzy quasi-orders)
  - **strongly right** and **left invariant** fuzzy equivalences (fuzzy quasi-orders)
  - **weakly right** and **left invariant** fuzzy equivalences (fuzzy quasi-orders)
- for any of these types of fuzzy relations we prove that **there is the greatest one**
- we give algorithms for computing these greatest relations
- for right and left invariant f.e. and f.q-o. the algorithms are iterative, and terminate after a finite number of steps if the underlying structure of truth values is **locally finite**
- but, they do not necessary terminate if this structure is not locally finite
- the greatest strongly right and left invariant f.e. and f.q-o. are **easier to compute** (without any iteration procedure), but they give **worse reductions**
- the greatest weakly right and left invariant f.e. and f.q-o. give **better reductions**, but they are **harder to compute**
- their computing includes determinization of the fuzzy recognizer or its reverse fuzzy recognizer, with the **possible blow up in the number of equations**

Better reductions

How to obtain better reductions?

- we have shown that fuzzy equivalences give better reductions than crisp ones
- we have shown that right and left invariant fuzzy quasi-orders give better reductions than right and left invariant fuzzy equivalences
- even better results in the state reduction can be obtained if we **alternate** reductions by means of the right type and left type fuzzy relations

Articles on state reduction of fuzzy automata

1. T. Petković, Congruences and homomorphisms of fuzzy automata, Fuzzy Sets and Systems 157 (2006) 444–458.
2. M. Ćirić, A. Stamenković, J. Ignjatović, T. Petković, Factorization of fuzzy automata, in: E. Csuhaj-Varju, and Z. Ésik (eds.), FCT 2007, Lecture Notes in Computer Science 4639 (2007) 213–225.
3. M. Ćirić, A. Stamenković, J. Ignjatović, T. Petković, Fuzzy relation equations and reduction of fuzzy automata, Journal of Computer and System Sciences (2009), doi:10.1016/j.jcss.2009.10.015.
4. A. Stamenković, M. Ćirić, J. Ignjatović, Reduction of fuzzy automata by means of fuzzy quasi-orders, Information Sciences (revised, resubmitted for publication).

Bisimulations and structural equivalence

Bisimulations for nondeterministic, fuzzy and weighted automata

- **Bisimulation** is a concept of structural equivalence that has shown oneself to be a very useful tool in many areas of computer science and mathematics, such as modal logic, concurrency theory, set theory, formal verification, model checking...
- they have been mostly used to model equivalence between states of the same system, and to reduce the number of states of this system.
- many algorithms have been proposed to compute the greatest bisimulation equivalence on a given labeled graph or a labeled transition system
- the faster ones are based on the crucial equivalence between the greatest bisimulation equivalence and the **relational coarsest partition problem**.
- Bisimulations between states of two distinct systems have been much less studied, probably due to lack of a suitable concept of a relation between two distinct sets which would behave like an equivalence
- the most often, bisimulations have been considered either as arbitrary relations (which have shown oneself to be too general), or as functions (which are too special)

Uniform bisimulations

Uniform bisimulations

- we have shown that bisimulations between two fuzzy automata can be successfully modeled by the concept of a **uniform fuzzy relation**
- we have defined and examined four types of uniform bisimulations: **forward**, **backward**, **forward-backward**, and **backward-forward**
- forward and backward bisimulation fuzzy equivalences between states of the same fuzzy automaton are just the right and left invariant fuzzy equivalences
- we have applied uniform bisimulations in the study of structural equivalence of fuzzy automata
- we also discuss related problems concerning bisimulations for nondeterministic and weighted automata.

Articles on bisimulations

1. M. Ćirić, J. Ignjatović, N. Damljanović, M. Bašić, Bisimulations for fuzzy automata, Fuzzy Sets and Systems (submitted).
2. M. Ćirić, J. Ignjatović, M. Bašić, I. Jančić, Nondeterministic automata: simulation, bisimulation, reduction, equivalence, Information and Computation (submitted).
3. N. Damljanović, M. Ćirić, J. Ignjatović, Bisimulations for weighted automata over additively idempotent semirings, (in preparation).

Fuzzy regular expressions

Fuzzy and weighted regular expressions

- **fuzzy regular expressions** have been introduced by Y. M. Li and W. Pedrycz [Fuzzy Sets and Systems 156 (2005) 68–92]
- they have shown that a fuzzy language can be represented by a fuzzy regular expression if and only if it can be recognized by a fuzzy finite recognizer
- they have not given an effective construction of a fuzzy finite recognizer which recognizes a fuzzy language represented by a given fuzzy regular expression
- we have given a generic method for constructing different types of fuzzy finite recognizers from fuzzy regular expressions
- viewing scalars appearing in a fuzzy regular expression as letters, this fuzzy regular expression can be considered as an ordinary regular expression over a new larger alphabet
- then this regular expression can be converted to a nondeterministic finite recognizer, using some of the well-known methods
- finally, we show that the obtained nondeterministic finite recognizer can be again transformed to a fuzzy finite recognizer over the original alphabet
- this fuzzy finite recognizer recognizes the fuzzy language represented by the original fuzzy regular expression

Position fuzzy recognizers

Position fuzzy recognizers of fuzzy regular expressions

- we have given an efficient method for computing the position fuzzy recognizer of a given fuzzy regular expression
- we study relationships between position fuzzy recognizers and other kinds of fuzzy finite recognizers obtained from fuzzy regular expressions, such as follow fuzzy recognizers and partial derivative fuzzy recognizers
- we have also discussed related problems concerning weighted regular expressions and weighted automata.

Articles on fuzzy regular expressions

1. A. Stamenković, M. Ćirić, Position fuzzy automata of fuzzy regular expressions, Fuzzy Sets and Systems (to be submitted).

Network analysis

Network analysis

- **network analysis** has originated a branch of sociology and mathematics which provides formal models and methods for the systematic study of social structures
- concepts of network analysis capture the common properties of **all networks** and its methods are applicable to the analysis of networks in general
- methods of network analysis are nowadays increasingly applied to many networks which are not social networks but share a number of commonalities with social networks
- such are the hyperlink structure on the Web, the electric grid, computer networks, information networks, or various large-scale networks appearing in nature
- network analysis takes a global view on network structures, based on the belief that types and patterns of relationships emerge from individual connectivity
- the presence (or absence) of such types and patterns have substantial effects on the network and its constituents
- the natural means to model networks mathematically is provided by the notions of graphs, relations and matrices
- methods of network analysis primarily originate from graph theory, semigroup theory and linear algebra

Fuzzy network analysis

Fuzzy network analysis

- relations between nodes in social and many other networks are often vague
- we have applied fuzzy approach to network analysis, studying **fuzzy networks**
- in the **positional analysis of networks**, whose aim is to find similarities between nodes, one of the the most studied notions is a **regular equivalence**
- two nodes are considered to be regularly equivalent if they are equally related to equivalent others
- we have examined **fuzzy regular equivalences**
- if a fuzzy network is viewed as a fuzzy automaton, these are are just those fuzzy equivalences which are both right and left invariant
- we have shown that we can get fuzzy regular equivalences as solutions of suitable systems of **fuzzy relation inequalities** and **fuzzy relation equations**
- we have given an algorithm for computing the greatest solution of these systems – **the greatest fuzzy regular equivalence**

Regular bisimulations

Regular bisimulations

- recall that regular fuzzy equivalences are used to capture similarity between nodes in a fuzzy network
- to capture similarity between nodes of **two different networks**, we have also introduced and studied the concept of a **regular bisimulation**

Articles on fuzzy network analysis

1. J. Ignjatović, M. Ćirić, S. Bogdanović, Fuzzy relation equations and positional analysis of fuzzy networks, International Journal of Approximate Reasoning (submitted).
2. I. Jančić, J. Ignjatović, M. Ćirić, Positional analysis of fuzzy networks and regular bisimulations, (in preparation).