Effective Clones

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Niš, November 2009

Outline

Clones

Local clones

Compact elements

Finitely generated clones

My two favorite clones

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Definition

Fix a set X (sometimes finite, usually countably infinite: $X = \mathbb{N}$). We write $\mathcal{O}^{(n)}$ for the set of n-ary operations: $\mathcal{O}^{(n)} = X^{X^n}$, and we let $\mathcal{O} = \mathcal{O}_X = \bigcup_{n=1,2,\ldots} \mathcal{O}^{(n)}$.

A clone on X is a set $C \subseteq \mathcal{O}$ which contains all the projection functions and is closed under composition.

Equivalently, a clone is the set of term functions of some universal algebra on X.

Fact

*The set of clones on X forms a complete Lattice: **CLONE**(X). (Order: $\subseteq$.)*

$\inf(C_1, C_2) = C_1 \cap C_2$.

$\sup(C_1, C_2) = \langle C_1 \cup C_2 \rangle$ — *the clone generated by $C_1 \cup C_2$.*

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Examples of clones

- Polynomial functions (in any sense of the word)
- Continuous operations on a topological space X
- All "essentially unary" operations: $f(x_1, \ldots, x_n) = F(x_i)$.
- All monotone operations (with respect to a fixed order $(X, \leq)$.)
- All operations mapping A^k into A (for fixed $A \subseteq X$).
- All operations "commuting" with $x \mapsto x + 1$: $f(x + 1, \ldots, z + 1) = f(x, \ldots, z) + 1$.
- ... many more ...

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Definition

A clone C is local if it is closed in the natural topology (product topology with X discrete). Equivalently: For all $g \notin C$ there is a finite witness for it.

Examples

- Polynomial functions. NO.
- Continuous functions. YES.
- All essentially unary functions. YES.
- All associative operations. NO. (associativity is not preserved in the limit)
- All operations preserving $\leq$ and $\neg$. YES.
- All operations commuting with $\neg$. YES.

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- Polynomial functions. NO.
- Cyclic maps. YES.

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- Continuous operations.
- Always only unary functions.
- Always constant operations. All constant operations.
- Always projections. Only projections.
- Always $\{0, 1\}$-valued functions.

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- Polynomial functions. NO.
- Continuous operations. NO.
- All essentially unary functions. YES.
- All operations generated by $x \cdot y$. YES.
- All operations preserving $x \cdot y$ with $x, y \in \{0, 1\}$. YES.
- All operations preserving $x \cdot y$. YES.

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- Polynomial functions. NO.
- Continuous operations. NO.
- All "essentially unary" functions.
- All operations generating all equivalence relations.
- All operations preserving x and y if $x \leq y$.
- All operations that only depend on finitely many arguments.

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- All operations essentially at most k ary. YES.
- All operations preserving ρ with arity k. YES.
- All operations preserving ρ. YES.

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- All monotone operations (with respect to $(X, \leq)$.) YES.
- All operations preserving a fixed equivalence relation. YES.

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The lattice of local clones

The set of clones on X forms a complete Lattice: **CLONE**(X).

Lattice operations: $C_1 \cap C_2$, $\langle C_1 \cup C_2 \rangle$.

Let **LOC**(X) be the set of local clones. **LOC**(X) is a suborder of **CLONE**(X), but not a sublattice:

Lattice operations: $C_1 \cap C_2$, $\langle C_1 \cup C_2 \rangle_{\mathrm{loc}}$,
where $\langle \cdots \rangle_{\mathrm{loc}}$ *is the local closure.*

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Proof: Let the set E of operations e such that e is a constant or a projection.

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All you need is **lub**

Definition

A complete lattice is a partial order $(L, \leq)$ in which every subset S has a least upper bound, $l.u.b.(S)$ or $\sup(S)$.
Every subset then also has a greatest lower bound $\inf(S)$.

Definition

Let L be a complete lattice. $x \in L$ is compact iff

$$\forall A \subseteq L : \sup A \geq x \Rightarrow \exists A_0 \subseteq A,\ A_0 \text{ finite},\ \sup A_0 \geq x.$$

Remark: A complete lattice in which every element is the supremum of compact elements is called "algebraic".

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Remark

If C_1 and C_2 are finitely generated, then $\langle C_1 \cup C_2 \rangle$ is also finitely generated, but $C_1 \cap C_2$ is in general not finitely generated.

Remark

Let $C = \langle E \rangle$ be finitely generated.

- *In case $E(x)$ is any, then C is finitely generated (trivially).*

Finitely generated clones

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Let $C = \langle E \rangle$ be finitely generated.

- *If the base set X is finite, then C is a decidable set (of finitary objects).*
- *If the base set X is infinite, then C is a recursively enumerable set (of finitary objects).*

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- *If the base set X is finite, then C is a decidable set (of finitary objects).*
- *If $X = \mathbb{N}$, then (even if E is a finite set of computable total functions) $\langle E \rangle$ and $\langle E \rangle_{tot}$ are in general not decidable, more precisely: $\{n \in N \mid \bar{n} \in \langle E \rangle\}$ may not be decidable (where $\bar{n}$ is the function with constant value n).*

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- *If the base set X is finite, then C is a decidable set (of finitary objects).*
- *If $X = \mathbb{N}$, then (even if E is a finite set of computable total functions) $\langle E \rangle$ and $\langle E \rangle_{\mathrm{loc}}$ are in general not decidable, more precisely: $\{n \in N \mid \bar{n} \in \langle E \rangle\}$ may not be decidable (where $\bar{n}$ is the function with constant value n).*

Finitely generated clones

Remark

If C_1 and C_2 are finitely generated, then $\langle C_1 \cup C_2 \rangle$ is also finitely generated, but $C_1 \cap C_2$ is in general not finitely generated.

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Finitely generated clones in **CLONE**(X)

Example

1. *Let $L =$ all subgroups of a fixed group G. Let $U \leq G$.*
   Then U is finitely generated iff U is compact in L.
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Finitely generated local clones

Theorem

Let $L := \mathbf{LOC}(X)$. Then the lattice L has no compact elements.

Question

Give a lattice-theoretic characterization of the finitely generated elements of $\mathbf{LOC}(X)$,
i.e., of $\{\, \langle E \rangle_{\mathrm{loc}} \mid E \subseteq X \text{ finite} \,\}$.

Remark: Not all local clones are finitely generated. Already on a finite set (where "local clone" = "clone") there are uncountably many clones, only countably many of them finitely generated.

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Outline

Clones

Local clones

Compact elements

Finitely generated clones

My two favorite clones

Almost unary functions

Let $X = \mathbb{N}$.

Definition

$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is "unary" iff there is a function $F : \mathbb{N} \to \mathbb{N}$ such that one of the following holds:

- $\forall x, y \, (f(x, y) = F(x))$
- $\forall x, y \, (f(x, y) = F(y))$

We call $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ "almost unary" iff there is a function $F : \mathbb{N} \to \mathbb{N}$ such that one of the following holds:

Let T_1 be the clone generated by all almost unary functions. (The binary functions in T_1 are exactly T_1.)

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Pinsker's Chain

For $k \geq 2$ let $m_k : \mathbb{N}^k \to \mathbb{N}$ be the following k-ary "generalized minimum":
$m_k(x_1, \ldots, x_k) = x_2$ whenever $x_1 \leq x_2 \leq \cdots \leq x_k$
$m_k(x_1, \ldots, x_k) = m(x_{\pi 1}, \ldots, x_{\pi k})$, whenever
$\pi : \{1, \ldots, k\} \to \{1, \ldots, k\}$ is a permutation.
For example, $m_2(x, y) = \max(x, y)$, and m_3 is the median function.

Theorem

Let $P_k := \langle T_1 \cup \{m_k\} \rangle$. Then all the P_k are distinct, $T_1 \subseteq \cdots \subseteq P_4 \subseteq P_3 \subseteq P_2 = \mathcal{O}$, and all clones containing T_1 are equal to one of the P_i.

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Gavrilov's theorem

Definition

$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is "somewhere 1-1" iff there are unary functions u, v such that one of the functions $(x, y) \mapsto f(u(x), v(y))$, $(x, y) \mapsto f(v(y), u(x))$ is 1-1 on the set $\{(x, y) \mid x < y\}$.
f is "never 1-1" if f is not "somewhere 1-1".

Let T_2 be the clone generated by the never 1-1 functions. (The binary functions in T_2 are exactly T_2.)

Theorem

T_1 and T_2 are the only clones C with the following properties: $C \neq \mathcal{O}$, C contains all unary functions, C is generated by binary functions, and
C is "maximal", i.e., $\langle C \cup \{b\} \rangle = \mathcal{O}$ for all binary $b \notin C$.

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An open question

T_1 is generated by all "almost unary" functions. T_1 is very nice:

- $T_1 = \{f(x_1, \ldots, x_n) \mid$ where f is the set of unary functions $\}$
- T_1 is countably many functions.

T_2 is generated by all "never 1-1" functions.
T_2 is very complicated:

An open question

T_1 is generated by all "almost unary" functions. T_1 is very nice:

- $T_1 = \langle U \cup \{p\} \rangle$ where U is the set of unary functions and p is a certain binary function
- T_1 is finitely generated
- We know all clones above T_1 (all of them)

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- T_1 is (not closed but) a Borel set
- We know all clones above T_1, quite effectively.

T_2 is generated by all "never 1-1" functions.
T_2 is very complicated:

An open question

T_1 is generated by all "almost unary" functions. T_1 is very nice:

- $T_1 = \langle U \cup \{p\} \rangle$ where U is the set of unary functions and p is a certain binary function
- T_1 is (not closed but) a Borel set
- We know all clones above T_1, quite effectively.

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- $T_2 \cap \mathcal{E}_2$ is anyone's guess

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- $T_2 \neq \langle U \cup E \rangle$ for any finite or countable set E
- T_2 has continuum many subclones

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- T_2 is not a Borel set
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