A logic for reasoning about time

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Overview

Goal: develop some formalism to be used in formal specification and verification of dynamic systems.

One possible solution: Region-based Propositional Logic of Time [with states] - **RPLT(S)**

In $RPLT(S)$ a dynamic system consists of

- states,
- evolution paths.

$RPLT(S)$ is modular:

$$RPLT(S) = \text{[basic region-based temporal reasoning]} \ (RPLT) + \text{[logic of states]}$$

The logic of states can be any kind of logic that is formalized as an institution.

Institutions

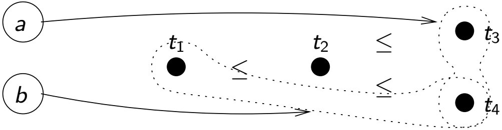
Institution[1]: a categorical abstract notion that formalizes the intuitive notion of a logical system.

$$\begin{tikzcd} & \mathbb{S}et \\ \mathbb{S}ig \arrow[ur, "\mathrm{Sen}"] \arrow[dr, "\mathrm{Mod}"'] & \models \\ & \mathbb{C}at^{\mathbf{op}} \end{tikzcd}$$

[1] J. Goguen and R. Burstall. Institutions: abstract model theory for specification and programming. Journal of the Association for Computing Machinery, 39, 95-146, 1992.

Semantics

Signature: a set V (whose elements are called symbols of primary temporal regions).
Temporal frame: a preordered set $(T, \leq)$.
Temporal model: a map $M : V \to \mathcal{P}(T)$.



Syntax

Temporal region: an element of the free Boolean algebra $\mathcal{B}(V)$ with operations:

- meet: $\sqcap$,
- join: $\sqcup$,
- complement: *,
- elements 0 and 1.

Atomic sentences:

- equality: $a = b$,
- weak temporal precedence: $a \prec_w b$,
- strong temporal precedence: $a \prec_s b$,

where $a, b \in \mathcal{B}(V)$.

Syntax

Sentences: by induction:

- any atomic sentence is a sentence,
- if φ and ψ are sentences then $\varphi \to \psi$ is a sentence,
- if φ is a sentence then $\neg\varphi$ is a sentence.

Secondary logical operators:

- $\varphi \vee \psi \stackrel{def}{=} \neg\varphi \to \psi$,
- $\varphi \wedge \psi \stackrel{def}{=} \neg(\varphi \to \neg\psi)$,
- $\varphi \leftrightarrow \psi \stackrel{def}{=} (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Satisfaction

Any model $M : V \to \mathcal{P}(T)$ has a unique homomorphic extension $M^{\#} : \mathcal{B}(V) \to \mathcal{P}(T)$:

$$M^{\#}(a \sqcup b) = M^{\#}(a) \cup M^{\#}(b), \quad M^{\#}(a \sqcap b) = M^{\#}(a) \cap M^{\#}(b),$$
$$M^{\#}(a^{*}) = T \setminus M^{\#}(a), \quad M^{\#}(0) = \emptyset, \quad M^{\#}(1) = T$$

For atomic sentences:

$M \models a = b$ if and only if $M^{\#}(a) = M^{\#}(b)$,

$M \models a \prec_w b$ if and only if $\forall y \in M^{\#}(b)$, $\exists x \in M^{\#}(a)$ such that $x \leq y$,

$M \models a \prec_s b$ if and only if $\forall y \in M^{\#}(b)$ and $\forall x \in M^{\#}(a)$, $x \leq y$.

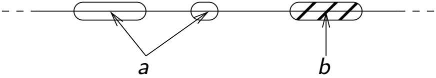
For more complex sentences:

$M \models \varphi \to \psi$ if and only if $M \models \varphi$ implies $M \models \psi$,

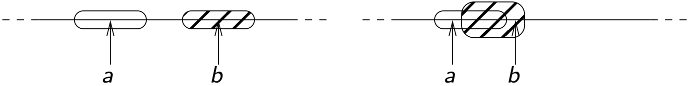
$M \models \neg\varphi$ if and only if $M \not\models \varphi$.

Some graphics

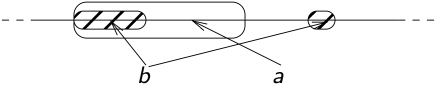
1. $a \sqcap b = 0$ (a and b are disjoint)



2. $a \prec_w b$ (a starts before b)

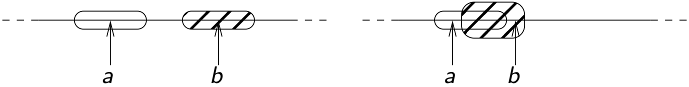


3. $(a \prec_w b) \wedge (b \prec_w a)$ (a and b start simultaneously)



Some graphics

2 $a \prec_w b$ (a starts before b)



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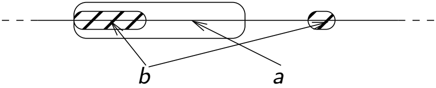
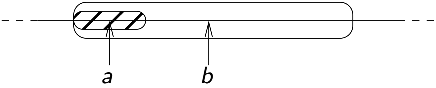


4 $(a \sqsubseteq b) \wedge (a \prec_w b)$ (a is a prefix of b)



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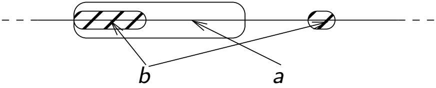

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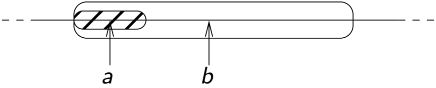

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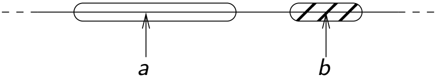

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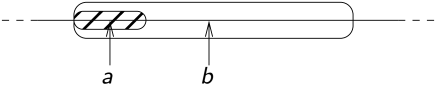

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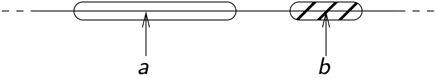
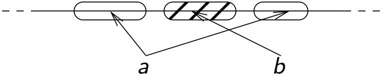
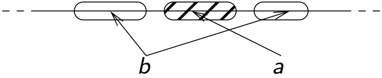


6 $(a \not\prec_t b) \wedge (b \not\prec_t a)$ (a and b are in a shuffle situation)





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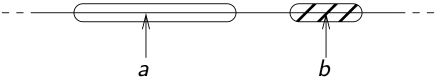

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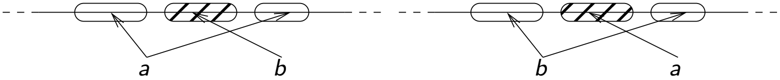
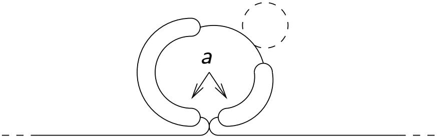


7 $(a \prec_t a)$ (a is part of a loop)



Adding states

Intuition:

- At each moment of time x we consider a snapshot $S(x)$ of the dynamic system,
  Snapshot: a structure specific to the logic of states.
- For two moments x and y such that $x \leq y$ we consider a homomorphic map of snapshots $S(x) \to S(y)$.
- A new type of sentence:

$<$ sentence specific to the logic of states $>$
is invariant relative to
$<$ a temporal region $>$.

Signatures

Let $\mathcal{I} = (\mathbb{S}ig^{\mathcal{I}}, \mathsf{Sen}^{\mathcal{I}}, \mathsf{Mod}^{\mathcal{I}}, \models^{\mathcal{I}})$ be an institution for the logic of states.

- $\mathcal{I}$-**temporal signature:** an ordered pair (V, Σ) where V is a *RPLT* signature and Σ is an $\mathcal{I}$-signature.
- **Homomorphism of** $\mathcal{I}$-**temporal signatures:** an ordered pair $(u, f) : (V, \Sigma) \to (V', \Sigma')$ such that $u : V \to V'$ is a function and $f : \Sigma \to \Sigma'$ is a homomorphism of $\mathcal{I}$-signatures.
- **Composition** of homomorphisms of $\mathcal{I}$-temporal signatures is done pairwise:

$$(u, f); (u', f') = (u; u', f; f')$$

Fact

Temporal signatures form a category:

$$\mathbb{T}\text{-}\mathbb{S}ig = \mathbb{S}ig^{RPLT} \times \mathbb{S}ig^{\mathcal{I}}$$

Models

Let (V, Σ) be a $RPLT(\mathcal{I})$-signature.
A (V, Σ)-**model** is a tuple $(T, \leq, M, S)$ where:

- $M : V \to \mathcal{P}(V)$ is a temporal model,
- $S : T \to \mathrm{Mod}^{\mathcal{I}}(\Sigma)$ is a functor.

Intuition:

- For a moment t, S_t is the snapshot at t,
- For two moments $s \leq t$, $S_{s\to t} : S_s \to S_t$ is the action that changes the state of the system from S_s into S_t.

$$\begin{tikzcd}
 & & t_3 & & & & S_{t_3} \\
t_1 \arrow[r, phantom, "\leq"] & t_2 \arrow[ur, phantom, "\leq"] \arrow[dr, phantom, "\leq"] & & \rightsquigarrow & S_{t_1} \arrow[r, "S_{t_1\to t_2}"] & S_{t_2} \arrow[ur, "S_{t_2\to t_3}"] \arrow[dr, "S_{t_2\to t_4}"'] & \\
 & & t_4 & & & & S_{t_4}
\end{tikzcd}$$

Model homomorphisms

A (V, Σ)-**model homomorphism** is a pair
$(h, \eta) : (T_1, \leq_1, M_1, S_1) \to (T_2, \leq_2, M_2, S_2)$ where:

- $h : M_1 \to M_2$ is a temporal model homomorphism,

$$h : T_1 \to T_2 \text{ is a monotonic map and}$$
$$h(M_1(t)) = M_2(t),\ \forall t \in T_1$$

- $\eta = \{\eta_t : (S_1)_t \to (S_2)_t\}_{t \in T_1}$ is a family of Σ-model homomorphisms natural in t.

$$\begin{tikzcd}
(S_1)_s \arrow[r, "\eta_s"] \arrow[d, "(S_1)_{s \to t}"'] \arrow[dr, phantom, "\circlearrowleft"] & (S_2)_{h(s)} \arrow[d, "(S_2)_{h(s) \to h(t)}"] \\
(S_1)_t \arrow[r, "\eta_t"'] & (S_2)_{h(t)}
\end{tikzcd}$$

Change of signature

Fact

(V, Σ)-models form a category: $\mathrm{T\text{-}Mod}(V, \Sigma)$.

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(V, Σ)-models form a category: $\text{T-Mod}(V, \Sigma)$.

Fact

Every homomorphism of $\mathcal{I}$-temporal signatures $(u, f) : (V, \Sigma) \to (V', \Sigma')$ *defines a functor* $\text{T-Mod}(u, f) : \text{T-Mod}(V', \Sigma') \to \text{T-Mod}(V, \Sigma)$.

▸ Proof sketch

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▸ Proof sketch

Fact

T-Mod *is a (contravariant) functorial map from* $\mathbb{T}$-$\mathbb{S}ig$ *to the dual of the quasi-category of categories,* $\mathbb{C}at^{op}$.

Sentences

(V, Σ)-**atomic sentences:**

- any V-atomic sentence (in *RPLT*) is a (V, Σ)-atomic sentence,
- if a is a temporal region and ρ is a Σ-sentence (in $\mathcal{I}$) then $a[\rho]$ is a (V, Σ)-atomic sentence.

(V, Σ)-**sentences:** by induction:

- any (V, Σ)-atomic sentence is a (V, Σ)-sentence,
- if φ and ψ are (V, Σ)-sentences then $\varphi \rightarrow \psi$ is a (V, Σ)-sentence,
- if φ is a (V, Σ)-sentence then $\neg\varphi$ is a (V, Σ)-sentence.

Change of signature

T-Sen(V, Σ): the set of (V, Σ)-sentences.

Fact

Every homomorphism of $\mathcal{I}$-temporal signatures $(u, f) : (V, \Sigma) \to (V', \Sigma')$ *defines a function* T-Sen(u, f) : T-Sen$(V, \Sigma) \to$ T-Sen(V', Σ').

▸ Proof sketch

Change of signature

T-Sen(V, Σ): the set of (V, Σ)-sentences.

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Every homomorphism of $\mathcal{I}$-temporal signatures $(u, f) : (V, \Sigma) \to (V', \Sigma')$ defines a function T-Sen(u, f) : T-Sen$(V, \Sigma) \to$ T-Sen(V', Σ').

▸ Proof sketch

Fact

T-Sen *is a functorial map from $\mathbb{T}$-$\mathbb{S}ig$ to the category of sets, $\mathbb{S}et$.*

Satisfaction

For (V, Σ)-atomic sentences:

$(T, \leq, M, S) \models a \bullet b$ if and only if $M \models a \bullet b$, for $\bullet \in \{=, \prec_w, \prec_s\}$,

$(T, \leq, M, S) \models a[\rho]$ if and only if $\forall t \in M^{\#}(a)$, $S_t \models^{\mathcal{I}} \rho$.

For more complex (V, Σ)-sentences:

$(T, \leq, M, S) \models \varphi \rightarrow \psi$ if and only if $(T, \leq, M, S) \models \varphi$ implies $(T, \leq, M, S) \models \psi$,

$(T, \leq, M, S) \models \neg\varphi$ if and only if $(T, \leq, M, S) \not\models \varphi$.

The satisfaction condition

Theorem

For any signature homomorphism $(u, f) : (V, \Sigma) \to (V', \Sigma')$*, any* (V', Σ')*-model* $(T', \leq', M', S')$ *and any* (V, Σ)*-sentence* φ*,*

$$\text{T-Mod}(u, f)(T', \leq', M', S') \models \varphi \quad \Leftrightarrow \quad (T', \leq', M', S') \models \text{T-Sen}(u, f)(\varphi)$$

Fact

$RPLT(\mathcal{I}) = (\mathbb{T}\text{-}\mathbb{S}ig, \text{T-Sen}, \text{T-Mod}, \models)$ *is an institution.*

Further work

- This paper presents a method that extends a logic formalized as an institution by adding a temporal dimension specific to the region-based propositional logic of time.
- The method is consistent with the notion of institution, thus the resulting logic can be studied within theory of institutions.

Some open questions:

- Can this method be further extended to an endofunctor of the category of institution (co)morphisms?
- What model theoretic properties of the base logic can be preserved through the process of adding a temporal dimension?

Thank you!

Change of signature

Fact

Every homomorphism of $\mathcal{I}$-temporal signatures $(u, f) : (V, \Sigma) \to (V', \Sigma')$ defines a functor $\text{T-Mod}(u, f) : \text{T-Mod}(V', \Sigma') \to \text{T-Mod}(V, \Sigma)$.

We define $\text{T-Mod}(u, f)$ by:

- $\text{T-Mod}(u, f)(T', \leq', M', S') = (T', \leq', u; M', S'; \text{Mod}^{\mathcal{I}}(f))$

$$V \xrightarrow{u} V' \xrightarrow{M'} \mathcal{P}(T)$$

$$T' \xrightarrow{S'} \text{Mod}^{\mathcal{I}}(\Sigma') \xrightarrow{\text{Mod}^{\mathcal{I}}(f)} \text{Mod}^{\mathcal{I}}(\Sigma)$$

- $\text{T-Mod}(u, f)(h', \eta') = (h', \eta' \text{Mod}^{\mathcal{I}}(f))$ where $\eta' \text{Mod}^{\mathcal{I}}(f)$ is

$$\{\text{Mod}^{\mathcal{I}}(f)(\eta'_{t'}) : \text{Mod}^{\mathcal{I}}(f)((S'_1)_{t'}) \to \text{Mod}^{\mathcal{I}}(f)((S'_2)_{h'(t')})\}_{t' \in T'_1}$$

Return

Change of signature

Fact

Every homomorphism of $\mathcal{I}$-temporal signatures $(u, f) : (V, \Sigma) \to (V', \Sigma')$ defines a function $\text{T-Sen}(u, f) : \text{T-Sen}(V, \Sigma) \to \text{T-Sen}(V', \Sigma')$.

We define $\text{T-Sen}(u, f)$ by:

- $\text{T-Sen}(u, f)(a \bullet b) = u(a) \bullet u(b)$ for $\bullet \in \{=, \prec_w, \prec_s\}$,
- $\text{T-Sen}(u, f)(a[\rho]) = u(a)[\text{Sen}^{\mathcal{I}}(f)(\rho)]$,
- $\text{T-Sen}(u, f)(\varphi \to \psi) = \text{T-Sen}(u, f)(\varphi) \to \text{T-Sen}(u, f)(\psi)$,
- $\text{T-Sen}(u, f)(\neg\varphi) = \neg\text{T-Sen}(u, f)(\varphi)$.

Return

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