Looking back and ahead: some problems in vertex-transitive graphs

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Hamilton cycles in vertex-transitive graphs

Semiregular automorphisms in vertex-transitive graphs

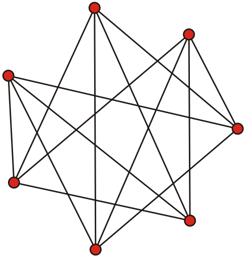
Hamilton cycles in vertex-transitive graphs

Vertex-transitive graphs & Cayley graphs

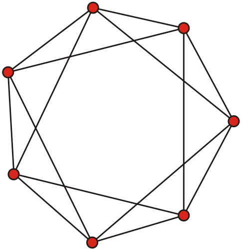
A graph $X = (V, E)$ is vertex-transitive if for any pair of vertices u, v there exists an automorphism α such that $\alpha(u) = v$.

A vertex-transitive graph is a Cayley graph provided it has a regular subgroup of automorphisms.

Vertex-transitive graphs



Is not VT



Is VT

Tying together two seemingly unrelated concepts: traversability and symmetry

Lovász question, '69

Does every connected vertex-transitive graph have a Hamilton path?

Lovász problem is, somewhat misleadingly, usually referred to as the Lovász conjecture, presumably in view of the fact that, after all these years, a connected vertex-transitive graph without a Hamilton path is yet to be produced.

VT graphs without Hamilton cycle

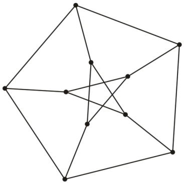
Only four connected VTG (having at least three vertices) not having a Hamilton cycle are known to exist:

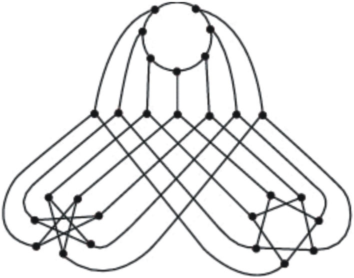
- the Petersen graph,
- the Coxeter graph,
- and the two graphs obtained from them by truncation.

All of these are cubic graphs, suggesting that no attempt to resolve the problem can bypass a thorough analysis of cubic VTG.

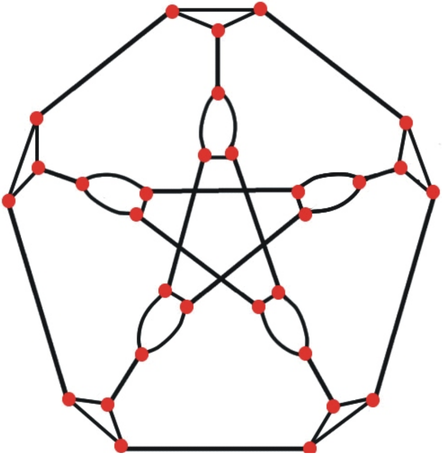
None of these four graphs is a Cayley graph, leading to the conjecture that every connected Cayley graph has a Hamilton cycle.

Vertex-transitive graphs without HC

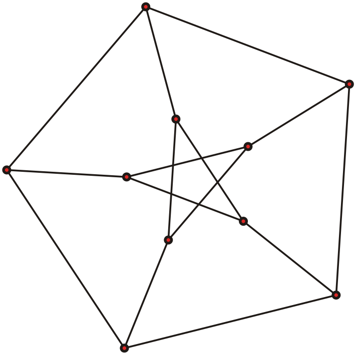




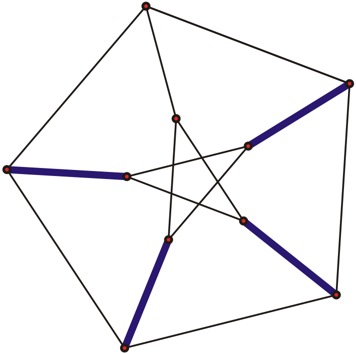
The truncation of the Petersen graph



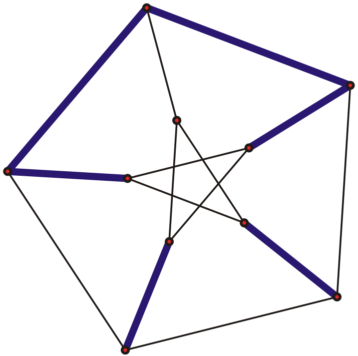
The Petersen graph



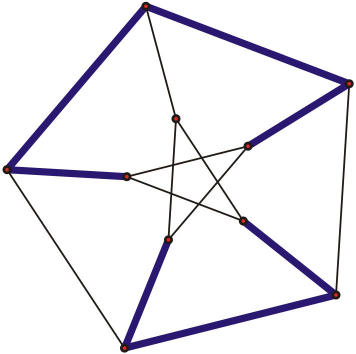
The Petersen graph



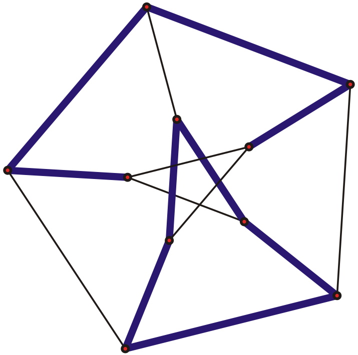
The Petersen graph



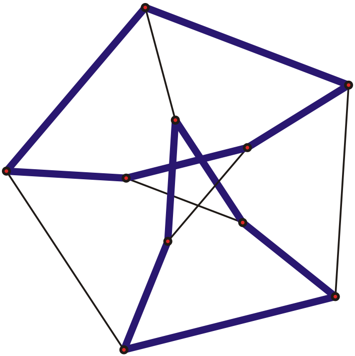
The Petersen graph



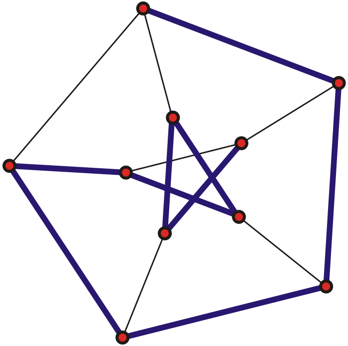
The Petersen graph



The Petersen graph



The Petersen graph



Hamiltonska pot

Conjectures/counterconjectures

Babai, '79

There exist infinitely many connected vertex-transitive graphs without a Hamilton cycle.

Thomassen, '91

There exist only finitely many such graphs.

The current situation

Hamiltonian cycles (paths) are known to exist in these cases:

- VTG of order kp ($k \leq 6$), $2p^2$, p^k (for $k \leq 4$) (Alspach, Chen, Du, Kutnar, Parsons, Šparl, Zhang, DM, etc.);
- CG of p-groups (Witte);
- VTG having groups with a cyclic commutator subgroup of order p^k (Durenberger, Gavlas, Keating, Morris, Morris-Witte, DM, etc.).
- CG $\mathrm{Cay}(G, \{a, b, a^b\})$, where a is an involution (Pak, Radoičić).
- Cubic CG $\mathrm{Cay}(G, S)$, where $S = \{a, b, c\}$ and $a^2 = b^2 = c^2 = 1$ and $ab = ba$ (Cherkassoff, Sjerve).
- Cubic CG $\mathrm{Cay}(G, S)$, where $S = \{x, y\}$ and $x^n = 1$, $y^2 = 1$ and $(xy)^3 = 1$ (Glover, Kutnar, DM).
- and in some other cases.

In short: the problem is still open.

The current situation

Hamilton cycles (paths) are known to exist for various families of Cayley graphs.

But not known whether they exist, e.g., for Cayley graphs of dihedral groups of order $2 \pmod 4$.

Hamiltonicity of vertex-transitive graphs

Essential ingredients in proof methods

- (Im)primitivity of transitive permutation groups.
- Existence of semiregular automorphisms in vertex-transitive graphs.
- Graph covering techniques.
- Graph embeddings.

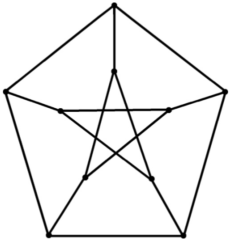
Semiregular automorphisms in vertex-transitive graphs

Semiregular automorphisms in VTG

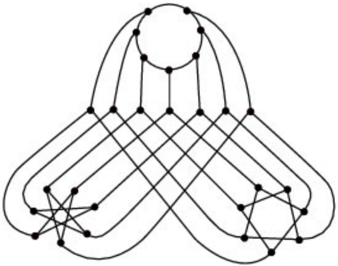
Does every vertex-transitive graph have a semiregular automorphism (DM, 1981; for transitive 2-closed groups, Klin, 1996)?

An element of a permutation group is semiregular, more precisely (m, n)-semiregular, if it has m orbits of size n and no other orbit.

Examples



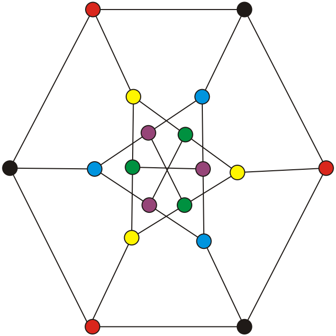
Has a (2,5)-semiregular automorphism

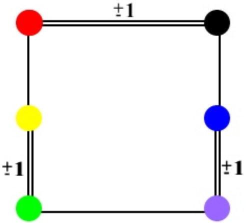


Has a (4,7)-semiregular automorphism

The Pappus graph

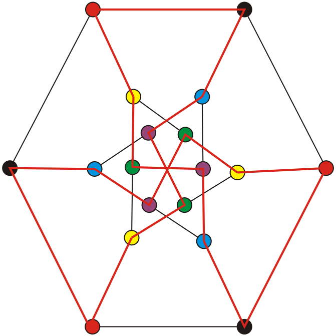
Hamiltonicity through semiregular elements

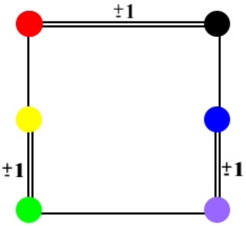




The Pappus graph

Hamiltonicity through semiregular elements





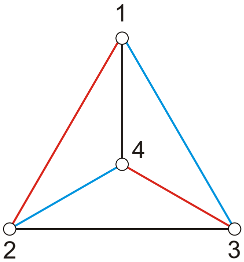
Semiregular elements

- (A) Automorphism groups of vertex-transitive (di)graphs;
- (B) 2-closed transitive permutation groups;
- (C) Transitive permutation groups.

Semiregular elements

(B) but not (A):

Regular action of $H = (\mathbb{Z}_2)^2 = \{id, (12)(34), (13)(24), (14)(23)\}$ on $V = \{1, 2, 3, 4\}$. Each of the orbital graphs has a dihedral automorphism group intersecting in H; so H is 2-closed but not the automorphism group of a (di)graph.



Semiregular elements

(C) but not (B):

$AGL(1, p^2)$, for $p = 2^k - 1$ a Mersenne prime, acting on the set of $p(p+1)$ lines of the affine plane $AG(2, p)$.

Results

- All transitive permutation groups of degree p^k or mp, for some prime p and $m < p$, have SE of order p (DM, '81).
- All cubic VTG have SA (DM, Scapellato, '93).
- All VTD of order $2p^2$ have SA of order p (DM, Scapellato, '93).
- All vertex-primitive graphs have SA (Giudici, '03).
- All vertex-quasiprimitive graphs have SA (Giudici, '03).
- All vertex-transitive bipartite graphs where only system of imprimitivity is the bipartition, have SA (Giudici, Xu, '07).
- Every 2-arc-transitive graph has SA (Xu, '07).
- Every ATG of prime valency has SA (Xu, '07).
- All quartic VTG have SA (Dobson, Malnič, DM, Nowitz, '07).

- All VTG of valency $p+1$ admitting a transitive $\{2, p\}$-group for p odd have SA (Dobson, Malnič, DM, Nowitz, '07).
- There are no elusive 2-closed groups of square-free degree (Dobson, Malnič, DM, Nowitz, '07).
- All ATG with valency pq, p, q primes, such that Aut(X) has a nonabelian minimal normal subgroup N with at least 3 vertex orbits, have SA (Xu, '08).
- Every VT, edge-primitive graph has SA (Giudici, Li, '09).
- All distance-transitive graphs have SA (Kutnar, Šparl, '09).

Semiregular automorphisms

The main steps towards a possible complete solution of the problem would have to consist of a proof of the existence of semiregular automorphisms in vertex-transitive graphs admitting a transitive solvable group.

Even for small valency graphs this is not easy. For example, valency 5 is still open.

Hvala!