Graph Covers

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November 10, 2009

Covering projections of graphs

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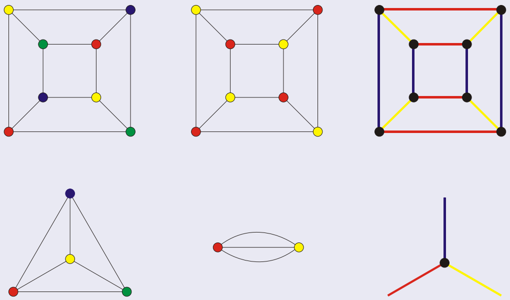
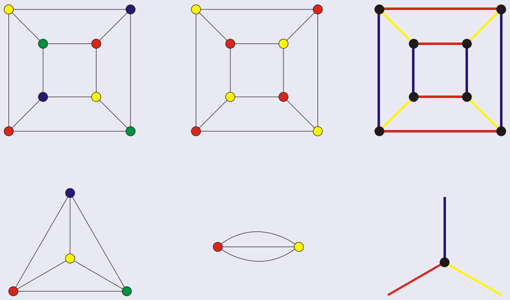


A surjective mapping $\mathrm{p}\colon \tilde{\mathrm{X}} \to \mathrm{X}$ such that $\mathrm{D}_{\tilde{\mathrm{u}}} \to \mathrm{D}_{\mathrm{u}}$ is a bijection



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$\tilde{\mathrm{X}}$: covering graph, X : base graph

$\mathrm{p}^{-1}(\mathrm{u}), \mathrm{p}^{-1}(\mathrm{x})$: fibres

Regular cover : fibres are orbits of a semiregular $\mathrm{CT}(\mathrm{p}) \leq \mathrm{Aut}\tilde{\mathrm{X}}$

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4. Systematic combinatorial theory of branched coverings of surfaces by Gross and Tucker 74'-77', Topological graph Theory 87'. Also by Jones and Singerman 78'.

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5. Vertex, edge, arc transitive, of small valencies

Conjecture of Marušič: every vertex transitive graph is a regular $\mathbb{Z}_{\mathrm{p}}$-cover of some smaller graph. Confirmed for valencies 3 and 4.

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5.3 Classifying graphs

Trans. graphs via orbital graphs (essentially: a trans. perm. group and its stabilizer). Djoković and Miller 80' classified stabilizers of arc trans. cubic graphs. Extended to edge-trans. cubic graphs by Goldschmidt 80'.

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Classify base graphs (usually using Classification of finite simple groups). The rest are normal covers. Can we give a neat description?

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All lifts of $G \leq \mathrm{Aut}X$ constitute a group $\tilde{G} \leq \mathrm{Aut}\tilde{X}$. There is a short exact sequence $1 \to \mathrm{CT}(p) \to \tilde{G} \to G \to 1$, and any abstract group extension can be studied as a lifting problem.

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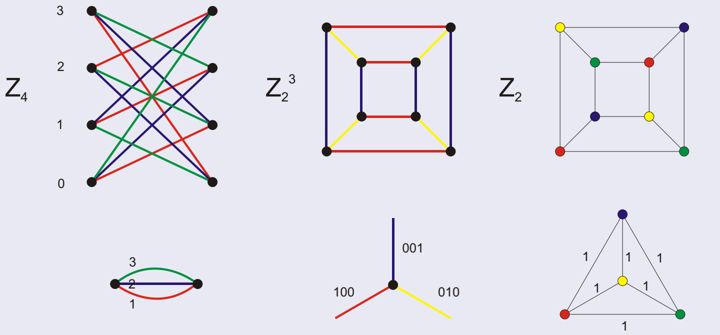
Djoković used topological approach via fundamental groups. Not appropriate to cope with specific problems arising with graphs.

Regular covers by regular voltages in $\Gamma \cong \mathrm{CT}(\mathrm{p})$



Covers and Lifting automorphisms combinatorially

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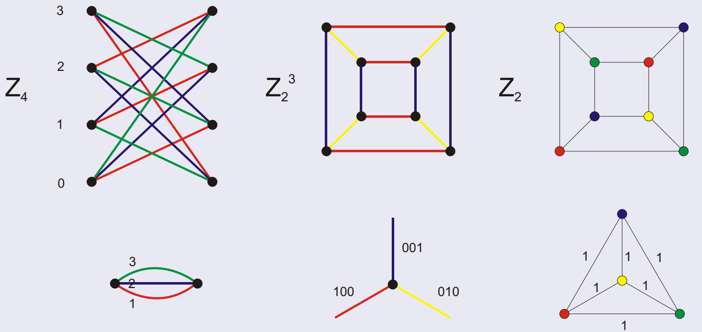

Basic lifting lemma (for regular covers)

$\mathrm{g} \in \mathrm{Aut}\mathrm{X}$ lifts $\Leftrightarrow$ the mapping $\mathrm{vol}(\mathrm{W}) \to \mathrm{vol}(\mathrm{W}^{\mathrm{g}})$ of voltages of the fundamental closed walks at some vertex extends to an automor. of Γ

Abelian regular covers

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For concrete covers one can use computer base packages like Magma for fast computation of invariant subspaces (MeatAxe algorithm).

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Minimal edge transitive subgroup cyclically permutes the edges, fixing the two vertices. Induced action on base cycles is given by

$$\mathrm{R} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \qquad \mathrm{R}^{\mathrm{t}} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

The full space gives rise to the homological cover. Proper notrivial invariant subspaces are 1-dim. $\Delta = \lambda^2 + \lambda + 1$, and the eigenvalues depend on the congruence type of $\mathrm{p} \mod 3$.

$\mathrm{p} = 3$, then $\lambda = 1$, and $\underline{\mathrm{v}} = [1, 2]^{\mathrm{t}}$
$\mathrm{p} \neq -1 \mod 3$, then Δ irreducible
$\mathrm{p} = 1 \mod 3$, then $\lambda = -\xi, -\xi^2$, and $\underline{\mathrm{v}} = [1, -\xi^2]^{\mathrm{t}}$, $[1, -\xi]^{\mathrm{t}}$

The eigenvectors induce voltages assignments on arcs, and hence determine the derived graphs. The obtained covers are pairwise nonequivalent, but the last two are isomorphic.

Classifying transitive covers of intransitive graphs

We consider regular abelain covers, and modified version of Praeger's normal reduction. Take quotients $\mathrm{X} \to \mathrm{X}/\mathrm{N}$ by normal subgroups arising from linear representation on eigenspaces of X.

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These subgroups can be computed in terms of eigenvalues and eigenvectors of the base graph, using complex irreducible characters of abelian voltage groups.

Thank you!